A portfolio optimality test based on the first-order stochastic dominance criterion

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Existing approaches to testing for the efficiency of a given portfolio make strong parametric assumptions about investor preferences and return distributions. Stochastic dominance based procedures promise a useful non-parametric alternative. However, these procedures have been limited to considering binary choices. In this paper we consider a new approach that considers all diversified portfolios, and thereby introduce a new concept of first-order stochastic dominance (FSD) optimality of a given portfolio relative to all possible portfolios. Using our new test, we show that the US stock market portfolio is significantly FSD non-optimal relative to benchmark portfolios formed on market capitalization and book-to-market equity ratios. Without appealing to parametric assumptions about the return distribution, we conclude that no nonsatiable investor would hold the market portfolio in the face of the attractive premia of small caps and value stocks.

I Introduction

Portfolio analysis and asset pricing tests typically focus on the mean-variance criterion. It is well-known that this criterion implicitly assumes a quadratic utility function or a normal probability distribution, which is quite restrictive in many cases. A good illustration of the limitations of the mean-variance criterion comes from (Levy (1998), p.2):"[Consider] two alternative investments: x providing \$1 or \$2 with equal probability and y providing \$2 or \$4 with equal probability, with an identical investment of, say, \$1.1. A simple calculation shows that both the mean and the variance of y are greater than the corresponding parameters of x; hence the mean-variance rule remains silent regarding the choice between x and y. Yet, any rational investor would (and should) select y, because the lowest return on y is equal to the largest return on x."

The criteria of stochastic dominance are useful non-parametric alternatives. Most notably, first-order stochastic dominance (FSD) is one of the basic concepts of decision making under uncertainty, relying only on the assumption of nonsatiation, or increasing utility. It does not require further specification of the shape of the utility function or the shape of the probability distribution. FSD analysis is generally more difficult to implement than mean-variance analysis. There exist well-known, simple tests for establishing FSD relationships between a pair of choice alternatives; see, for example, (Levy (1998), Section 5.2). Unfortunately, these tests have limited use for portfolio analysis and asset pricing tests, because investors generally can form a large number of portfolios by diversifying across individual assets. Therefore, there is a need to develop a test for establishing if a given portfolio is "FSD efficient" relative to all possible portfolios. Such a test would be a useful alternative for existing mean-variance portfolio efficiency tests (for example, Gibbons, Ross and Shanken (1989)), especially if the return distribution is skewed and fat-tailed.

A complication in testing FSD portfolio efficiency is that we must distinguish between efficiency criteria based on "admissibility" and "optimality". There is a subtle difference between these two concepts. A choice alternative is FSD admissible if and only if no other alternative is preferred by all nonsatiable decision-makers. A choice alternative is FSD optimal if and only if it is the optimal choice for at least some nonsatiable decision-maker. For pairwise comparison, the two concepts are identical; alternative x_1 is FSD undominated by alternative x_2 if and only if some nonsatiable decision-maker prefers x_1 to x_2. However, more generally, when multiple choice alternatives are available, FSD admissibility is a necessary but not sufficient condition for FSD optimality. In other words, a choice alternative may be admissible even if it is not optimal for any increasing utility function.

Bawa *et al.* (1985) and Kuosmanen (2004) propose FSD tests that apply under more general conditions than a pairwise test does. The two tests differ in a subtle way. While Bawa *et al.* (1985) consider all convex combinations of the distribution functions of a given set of choice alternatives, Kuosmanen considers the distribution function for all convex combinations of a given set of choice alternatives. Each of these two tests captures an important aspect of portfolio choice that is not captured by a pairwise FSD test. Still, both tests miss some key aspect of a proper FSD portfolio optimality test and both tests generally give a necessary but not sufficient condition. The linear programming test of Bawa *et al.* is based on optimality, but it does not account for full diversification across the choice alternatives. Bawa *et al.* use a set of undiversified base assets as the choice alternatives. In principle, diversification can enter through the back door by including combinations of the base assets as additional choice alternatives. However, since the number of possible combinations is infinitely large, this approach generally gives only a necessary condition and it yields a potentially very large computation load. The mixed integer linear programming test of Kuosmanen does account for full diversification, but it relies on admissibility rather than optimality.

In this study, we derive a proper test for FSD optimality of a given portfolio relative to all portfolios formed from a set of choice alternatives and apply that test to analyze the US stock market portfolio. In contrast to Bawa *et al.* (1985), our test considers all diversified portfolios in addition to the individual, undiversified choice alternatives, and in contrast to Kuosmanen (2004), it relies on optimality rather than admissibility. Both features lead to a more powerful FSD test, based on a necessary and sufficient condition, than is currently available.

The new test contributes to recent methodological developments that make the stochastic dominance methodology more applicable to problems in financial economics by improving the statistical power and providing more efficient computation algorithms. Our test is a natural complement to the second-order stochastic dominance (SSD) efficiency test of Post (2003). Due to concavity of utility, the analysis of SSD is generally simpler than that of FSD. First, SSD admissibility and SSD optimality are equivalent in a portfolio context and the definition of "SSD efficiency" is less ambiguous than that of "FSD efficiency".[2] Second, SSD efficiency can be tested by simply evaluating the first-order optimality condition for all individual, undiversified choice alternatives. Third, the representative utility functions have a piecewise-linear shape and the first-order optimality condition can be checked by searching over these functions using a single small-scale linear programming problem.

We apply our test to US stock market data in order to analyze the FSD optimality of the market portfolio relative to portfolios formed on market capitalization and book-to-market equity ratio. This application seems relevant because a large class of capital market equilibrium models predict that the market portfolio is FSD optimal. Surprisingly, we find that the market portfolio is significantly FSD non-optimal. Without appealing to parametric assumptions about the return distribution, we conclude that no nonsatiable investor would hold the market portfolio in the face of the attractive premia of small caps and value stocks.

The remainder of this text is structured as follows. Section II introduces preliminary notation, assumptions and definitions. Next, Section III reformulates the FSD optimality criterion in terms of piecewise-constant representative utility functions, in the spirit of the representative utility functions used by Russell and Seo (1989). Section IV develops a linear programming test for searching over all representative utility functions in order to test portfolio optimality and suggests several approaches to identifying the input to this test. Section V uses a numerical example to illustrate our test and compare it with the two existing tests. Section VI discusses our empirical analysis of the US stock market portfolio. Finally, Section VII presents concluding remarks and suggestions for further research.

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[2]Theorem 1 of Post (2003) shows the equivalence using Sion's (1958) Minimax Theorem. Other treatments of SSD admissibility and optimality include Peleg and Yaari (1975), Dybvig and Ross (1982), and Bawa and Goroff (1982, 1983).

II Preliminaries

Consider N choice alternatives and T scenarios with equal probability. The outcomes of the choice alternatives in the various scenarios are given by

$$X = \begin{pmatrix} \mathbf{x}^1 \\ \mathbf{x}^2 \\ \vdots \\ \mathbf{x}^T \end{pmatrix}$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_N^t)$ is the t-th row of matrix X. Without loss of generality we can assume that the columns of X are linearly independent. In addition to the individual choice alternatives, the decision-maker may also combine the choice alternatives into a portfolio. We will use $\boldsymbol{\lambda} \in R^N$ for a vector of portfolio weights and the portfolio possibilities are given by

$$\Lambda = \{\boldsymbol{\lambda} \in R^N | \mathbf{1}'\boldsymbol{\lambda} = 1, \ \lambda_n \geq 0, \ n = 1, 2, \dots, N\}.^3$$

The evaluated portfolio is denoted by $\boldsymbol{\tau} \in \Lambda$ and is assumed to be risky [4]. Let $y^{[k]}$ be the k-th smallest element among $y^1, y^2, ..., y^N$, that is, $y^{[1]} \leq y^{[2]} \leq \dots \leq y^{[N]}$. Let

$$\underline{m} = \min_{t,n} x_n^t, \quad \overline{m} = \max_{t,n} x_n^t \quad \text{and} \quad k(\boldsymbol{\tau}) = \min\{t : (X\boldsymbol{\tau})^{[t]} > (X\boldsymbol{\tau})^{[1]}\}.$$

The constants $\underline{m}$ and $\overline{m}$ are the minimum and maximum possible return. After ordering the returns of the tested portfolio $\boldsymbol{\tau}$ from the smallest to the largest one, $k(\boldsymbol{\tau})$ determines the order of the second smallest return. Without ties, we have $k(\boldsymbol{\tau}) = 2$, but if the smallest value occurs multiple times, then $k(\boldsymbol{\tau}) > 2$.

Decision-makers obey to the rules of expected utility theory. Their preferences belong to the class of weakly increasing utility functions U_1 and their decision-making problem can be represented as

$$(1) \qquad \max_{\boldsymbol{\lambda} \in \Lambda} \sum_{t=1}^{T} u(\mathbf{x}^t \boldsymbol{\lambda}).$$

[3] By using the simplex Λ , we exclude short selling. Short selling typically is difficult to implement in practice due to margin requirements and explicit or implicit restrictions on short selling for institutional investors. Still, we may generalize our analysis to include (bounded) short selling. In fact, the analysis applies to any portfolio set that takes the form of a polytope (roughly speaking, a non-empty and closed set that is defined by linear restrictions) if we replace the N choice alternatives with the set of M extreme points of the polytope.

[4] Testing optimality for a riskless portfolio is trivial, because we then only need to check if there exists some portfolio that achieves a higher minimum return than the riskless rate. If no such portfolio exists, the riskless alternative is the optimal solution for extreme risk averters and hence FSD optimal.

Since utility functions are unique up to the level of a positive linear transformation, without loss of generality, we may focus on the following set of standardized utility functions:

$$U_1(\boldsymbol{\tau}) \;=\; \{u \in U_1 : u(\underline{m}) = 0;\;\; u((X\boldsymbol{\tau})^{[T]}) - u((X\boldsymbol{\tau})^{[k(\boldsymbol{\tau})]}) = 1\}. \tag{2}$$

Note that the standardization depends on the evaluated portfolio and hence will differ for evaluating different portfolios. Furthermore, the standardization requires utility to be strictly increasing at least somewhere in the interior of the range for the evaluated portfolio. This requirement is natural, because, testing optimality relative to all $u \in U_1$ is trivial. Specifically, every portfolio $\boldsymbol{\lambda} \in \Lambda$ is an optimal solution for $u_0 = I(x \geq (X\boldsymbol{\tau})^{[1]})$, that is, two-piece constant utility function. Thus $U_1(\boldsymbol{\tau})$ is the largest subset of U_1 for which testing optimality is non-trivial.

Definition 1:

Portfolio $\boldsymbol{\tau} \in \Lambda$ is FSD optimal if and only if it is the optimal solution of (1) for at least some utility function $u \in U_1(\boldsymbol{\tau})$, that is, there exists $u \in U_1(\boldsymbol{\tau})$ such that

$$\sum_{t=1}^{T} u(\mathbf{x}^t\boldsymbol{\tau}) - \sum_{t=1}^{T} u(\mathbf{x}^t\boldsymbol{\lambda}) \geq 0 \quad \forall \boldsymbol{\lambda} \in \Lambda.$$

Otherwise, $\boldsymbol{\tau}$ is FSD non-optimal.

The intuition behind FSD optimality is that the evaluated portfolio is of potential interest to investors if it achieves a higher expected utility than all other portfolios for some increasing utility function. This concept allows for several variations. Most notably, we can choose between weakly and strictly increasing utility and we can choose between weakly and strongly higher expected utility. Empirically, these variations are often not distinguishable. A weakly increasing utility function $u(x)$ generally is empirically indistinguishable from the strictly increasing function $u(x) + ax$ for some infinitely small value $a > 0$. Similarly, infinitely small data perturbations generally suffice to change a weak inequality to a strong one. In addition, it can be shown that requiring strictly increasing utility and strong inequality is the same as weakly increasing utility and weak inequality. This study will not try to answer the question which type of utility function or inequality is most relevant. Rather, we will focus on accounting for all possible portfolios in an optimality test that is based on weakly increasing utility and weak inequality.

III Representative utility functions

This section reformulates the optimality criterion in terms of a set of elementary representative utility functions. For pairwise FSD comparisons, Russell and Seo (1989) show that the set of three-piece linear utility functions is representative for all admissible utility functions. In our portfolio context, with diversification allowed, a class of piecewise constant utility functions is relevant:

$$(3)\; R_1(\boldsymbol{\tau}) = \{u \in U_1 | u(y) = \sum_{t=1}^{T} a_t I(y \geq (X\boldsymbol{\tau})^{[t]}), \quad \mathbf{a} \in A(\boldsymbol{\tau})\}$$

$$(4)\; A(\boldsymbol{\tau}) = \{\mathbf{a} \in \mathbb{R}_+^T : \sum_{t=k(\boldsymbol{\tau})}^{T} a_t = 1, \quad (X\boldsymbol{\tau})^{[t]} = (X\boldsymbol{\tau})^{[s]} \;\wedge\; t < s \Rightarrow a_s = 0, \quad t, s = 1, 2, \ldots, T\}$$

where

$$I(y \geq y_0) = 1 \text{ for } y \geq y_0$$
$$= 0 \text{ otherwise.}$$

This class consists of at most $(T + 1)$ - piece constant, upper semi-continuous utility functions. This class is reminiscent of the representative utility functions used by Russell and Seo (1989) to test pairwise FSD relationship. In fact, our utility functions can be obtained as a sum of the first derivatives of the Russell and Seo (1989) representative utility functions on the relevant interval $(\underline{m}, \overline{m})$.[5] The utility functions are also reminiscent of the piecewise linear functions used by Post (2003) to test SSD portfolio efficiency.

Theorem 1:

Portfolio $\boldsymbol{\tau} \in \Lambda$ is FSD optimal if and only if it is the optimal solution of (1) for at least some utility function $u \in R_1(\boldsymbol{\tau})$, that is, there exists $u \in R_1(\boldsymbol{\tau})$ such that

$$\sum_{t=1}^{T} u(\mathbf{x}^t \boldsymbol{\tau}) - \sum_{t=1}^{T} u(\mathbf{x}^t \boldsymbol{\lambda}) \geq 0 \quad \forall \boldsymbol{\lambda} \in \Lambda.$$

Otherwise, $\boldsymbol{\tau}$ is FSD non-optimal.

[5]Russell and Seo (1989) functions are continuous three-piece functions that consist of two constant pieces and one linear, increasing piece in between. Choose T such functions with increasing pieces with slopes $a_1, a_2, ..., a_T$ for the intervals $((X\tau)^{[1]}, (X\tau)^{[2]})$, $((X\tau)^{[2]}, (X\tau)^{[3]})$,...,$((X\tau)^{[T-1]}, (X\tau)^{[T]})$, $((X\tau)^{[T]}, \overline{m})$. Our piecewise constant utility function is the sum of the first derivatives on these intervals.

Proof:

The sufficient condition follows directly from $R_1(\boldsymbol{\tau}) \subset U_1(\boldsymbol{\tau})$. To establish the necessary condition, suppose that $\boldsymbol{\tau}$ is optimal for $u(y) \in U_1(\boldsymbol{\tau})$ and let

$$u_R(y) = \sum_{t=1}^{T} a_t I(y \geq (X\boldsymbol{\tau})^{[t]}),$$

with $a_1 = u(X\boldsymbol{\tau})^{[1]}$, $a_t = 0$, $t = 2,\ldots,k(\boldsymbol{\tau}) - 1$ and $a_t = u(X\boldsymbol{\tau})^{[t]} - u(X\boldsymbol{\tau})^{[t-1]}$, $t = k(\boldsymbol{\tau}),\ldots,T$. By construction, $u_R(y) \in R_1(\boldsymbol{\tau})$. Furthermore, $u_R(y) \leq u(y)$, $\forall y \in \langle \underline{m}, \overline{m} \rangle$ and $u_R(y) = u(y)$, for $y = (X\boldsymbol{\tau})^{[1]}, (X\boldsymbol{\tau})^{[2]}, \ldots, (X\boldsymbol{\tau})^{[T]}$. Therefore,

$$\sum_{t=1}^{T} u_R(\mathbf{x}^t\boldsymbol{\tau}) - \sum_{t=1}^{T} u_R(\mathbf{x}^t\boldsymbol{\lambda}) \geq \sum_{t=1}^{T} u(\mathbf{x}^t\boldsymbol{\tau}) - \sum_{t=1}^{T} u(\mathbf{x}^t\boldsymbol{\lambda}) \qquad \forall \boldsymbol{\lambda} \in \Lambda.$$

Since $\boldsymbol{\tau}$ is optimal for $u(y) \in U_1(\boldsymbol{\tau})$, the RHS is nonnegative for all $\boldsymbol{\lambda} \in \Lambda$, and hence $\boldsymbol{\tau}$ is also optimal for $u_R(y) \in R_1(\boldsymbol{\tau})$, which completes the proof. □

The proof makes use of the fact that any utility function can be transformed into a piecewise constant function with increments only at $\mathbf{x}^t\boldsymbol{\tau}$, $t = 1,\ldots,T$. This transformation does not affect the expected utility for the evaluated portfolio but it may lower the expected utility of other portfolios. Since the objective is to analyze if the evaluated portfolio is optimal for some utility function, only the representative utility functions need to be checked; all other utility functions are known to put the evaluated portfolio in a worse perspective than some representative utility function.

To illustrate the representation theorem, consider the cubic utility function $u(y) = 10 + y - 0.1y^2 + 0.05y^3$ and a portfolio with returns $(X\boldsymbol{\tau})^{[1]} = -5$, $(X\boldsymbol{\tau})^{[2]} = 1$ and $(X\boldsymbol{\tau})^{[3]} = 6$. Figure 1 shows a version of this function that is transformed such that it belongs to $U_1(\boldsymbol{\tau})$: $u_0(y) = 2.6 + 0.04y - 0.004y^2 + 0.002y^3$ (the solid line). Since the latter function is obtained after a positive linear transformation, it yields the same results as the former function. The dashed line gives the piecewise-constant function $u_R(y) = 2.087I(y \geq -5) + 0.546I(y \geq 1) + 0.454I(y \geq 6)$. This function is constructed such that it yields exactly the same utility levels for the evaluated portfolio as $u_0(y)$ does. Furthermore, the utility levels for all other portfolios are smaller than or equal to those for $u_0(y)$. Thus, if the evaluated portfolio is optimal for $u_0(y)$, then it is also optimal for $u_R(y)$. A similar analysis applies for every admissible utility function $u(y) \in U_1(\boldsymbol{\tau})$.

[Insert Figure 1 about here]

Apart from replacing $U_1(\boldsymbol{\tau})$ with $R_1(\boldsymbol{\tau})$, we may also replace Λ with a reduced portfolio set that considers only portfolios with a higher minimum than the evaluated portfolio:

$$\Lambda(\boldsymbol{\tau}) = \left\{ \boldsymbol{\lambda} \in \Lambda : (X\boldsymbol{\tau})^{[1]} \leq (X\boldsymbol{\lambda})^{[1]} \right\}.$$

Using the representative utility functions and the reduced portfolio set, we can construct the following FSD non-optimality measure for any $\Lambda_0 \subseteq \Lambda(\boldsymbol{\tau})$:

$$\xi(\boldsymbol{\tau}, \Lambda_0) = \frac{1}{T} \min_{u \in R_1(\boldsymbol{\tau})} \max_{\boldsymbol{\lambda} \in \Lambda_0} \sum_{t=1}^{T} \left(u(\mathbf{x}^t \boldsymbol{\lambda}) - u(\mathbf{x}^t \boldsymbol{\tau}) \right). \tag{5}$$

Replacing Λ with $\Lambda(\boldsymbol{\tau})$ reduces the parameter space but it causes no harm, because

$$\max_{\boldsymbol{\lambda} \in \Lambda} \sum_{t=1}^{T} \left(u(\mathbf{x}^t \boldsymbol{\lambda}) - u(\mathbf{x}^t \boldsymbol{\tau}) \right) = \max_{\boldsymbol{\lambda} \in \Lambda(\boldsymbol{\tau})} \sum_{t=1}^{T} \left(u(\mathbf{x}^t \boldsymbol{\lambda}) - u(\mathbf{x}^t \boldsymbol{\tau}) \right)$$

for all $u \in R_1(\boldsymbol{\tau})$ with sufficiently large a_1 and we minimize the maximum of expected utility differences. If the evaluated portfolio has the highest minimum then we can directly conclude that $\xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau})) = 0$, that is, the evaluated portfolio is FSD optimal (see the following Corollary).

Corollary 1:

(i) Portfolio $\boldsymbol{\tau}$ *is FSD optimal if and only if* $\xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau})) = 0$. *Otherwise,* $\xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau})) > 0$.

(ii) If $\Lambda_0 \subseteq \Lambda(\boldsymbol{\tau})$ *then* $\xi(\boldsymbol{\tau}, \Lambda_0) \leq \xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau}))$.

The next section will show that $\xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau}))$ can be computed by solving a linear programming problem.

IV Mathematical Programming Algorithm

There exist well-known, simple algorithms for establishing FSD-dominance relationships between a pair of choice alternatives; see, for example, (Levy (1998), Section 5.2). Bawa *et al.* (1985) derive a linear programming algorithm for FSD optimality relative to a discrete set of choice alternatives. Kuosmanen's (2004) test for FSD admissibility in a portfolio context is computationally more demanding, because we need to account for changes to the ranking of the portfolio returns as the portfolio weights change, a task that requires integer programming. A similar complication arises for testing FSD optimality in a portfolio context. This section develops a linear programming test for testing portfolio optimality. However, the input to the linear programming test may require an initial phase of mixed integer linear programming (MILP) or subsampling.

Before presenting the algorithm, we stress that in some cases, simple necessary or sufficient conditions will suffice to classify the evaluated portfolio as FSD optimal or FSD non-optimal. For example, a pairwise dominance relationship or a non-optimality classification by the Bawa *et al.* suffice to conclude that the portfolio is FSD non-optimal. Similarly, if the evaluated portfolio is classified as efficient according to a mean-variance test or a SSD test, we can conclude that the portfolio is FSD optimal.

Let

$$h_s(\boldsymbol{\lambda},\boldsymbol{\tau}) = \sum_{t=1}^{T} I(\mathbf{x}^t\boldsymbol{\lambda} \geq (X\boldsymbol{\tau})^{[s]}), \qquad s = 1,\dots,T \tag{6}$$

$$\mathbf{h}(\boldsymbol{\lambda},\boldsymbol{\tau}) = (h_1(\boldsymbol{\lambda},\boldsymbol{\tau}),\dots,h_T(\boldsymbol{\lambda},\boldsymbol{\tau})) \tag{7}$$

$$H(\boldsymbol{\tau}) = \{\mathbf{h} \in \{0,\dots,T\}^T : \mathbf{h} = \mathbf{h}(\boldsymbol{\lambda},\boldsymbol{\tau}), \quad \boldsymbol{\lambda} \in \Lambda(\boldsymbol{\tau})\}. \tag{8}$$

Since $h_s(\boldsymbol{\lambda},\boldsymbol{\tau})$ represents the number of returns of portfolio $\boldsymbol{\lambda}$ exceeding the s-th smallest return of portfolio $\boldsymbol{\tau}$, it can take at most $T+1$ values $(0,1,\dots,T)$ for any $s = 1,\dots,T$. Thus the set $H(\boldsymbol{\tau})$ has a finite number of elements. For small-scale applications, identifying all elements is a fairly trivial task. However, for large-scale applications, the task is more challenging and can become computationally demanding. Some computational strategies to identifying the elements of $H(\boldsymbol{\tau})$ are discussed below. Interestingly, given $H(\boldsymbol{\tau})$, the test statistic $\xi(\boldsymbol{\tau},\Lambda(\boldsymbol{\tau}))$ can be computed using simple linear programming. To see this, consider the following chain of equalities:

$$\begin{aligned}
\xi(\boldsymbol{\tau},\Lambda(\boldsymbol{\tau})) &= \frac{1}{T}\min_{u\in R_1(\boldsymbol{\tau})}\max_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{\tau})}\sum_{t=1}^{T}\left(u(\mathbf{x}^t\boldsymbol{\lambda}) - u(\mathbf{x}^t\boldsymbol{\tau})\right)\\
&= \frac{1}{T}\min_{\mathbf{a}\in A(\boldsymbol{\tau})}\max_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{\tau})}\sum_{t=1}^{T}\sum_{s=1}^{T} a_s\left(I(\mathbf{x}^t\boldsymbol{\lambda} \geq (X\boldsymbol{\tau})^{[s]}) - I(\mathbf{x}^t\boldsymbol{\tau} \geq (X\boldsymbol{\tau})^{[s]})\right)\\
&= \frac{1}{T}\min_{\mathbf{a}\in A(\boldsymbol{\tau})}\max_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{\tau})}\sum_{t=1}^{T}\sum_{s=k(\boldsymbol{\tau})}^{T} a_s\left(I(\mathbf{x}^t\boldsymbol{\lambda} \geq (X\boldsymbol{\tau})^{[s]}) - I(\mathbf{x}^t\boldsymbol{\tau} \geq (X\boldsymbol{\tau})^{[s]})\right)\\
&= \frac{1}{T}\min_{\mathbf{a}\in A(\boldsymbol{\tau})}\max_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{\tau})}\sum_{s=k(\boldsymbol{\tau})}^{T} a_s\left(\sum_{t=1}^{T} I(\mathbf{x}^t\boldsymbol{\lambda} \geq (X\boldsymbol{\tau})^{[s]}) - \sum_{t=1}^{T} I(\mathbf{x}^t\boldsymbol{\tau} \geq (X\boldsymbol{\tau})^{[s]})\right)\\
&= \frac{1}{T}\min_{\mathbf{a}\in A(\boldsymbol{\tau})}\max_{\boldsymbol{\lambda}\in\Lambda(\boldsymbol{\tau})}\sum_{s=k(\boldsymbol{\tau})}^{T} a_s(h_s(\boldsymbol{\lambda},\boldsymbol{\tau}) - h_s(\boldsymbol{\tau},\boldsymbol{\tau}))
\end{aligned}$$

$$= \frac{1}{T} \min_{\mathbf{a} \in A(\boldsymbol{\tau}), \delta} \left\{ \delta : \sum_{s=k(\boldsymbol{\tau})}^{T} a_s (\overline{h}_s - h_s(\boldsymbol{\tau}, \boldsymbol{\tau})) \leq \delta \quad \forall \overline{\mathbf{h}} \in H(\boldsymbol{\tau}) \right\}$$

The RHS of the final equality involves the minimization of a linear objective under a finite set of linear constraints. Thus, testing FSD optimality requires solving a simple linear programming problem and Corollary 1(i) implies the following sufficient and necessary condition for FSD optimality.

Theorem 2:

Let $H_0 \subseteq H(\boldsymbol{\tau})$. Let

$$\delta^*(H_0) = \min_{\mathbf{a} \in A(\boldsymbol{\tau})} \quad \delta \tag{9}$$

$$\text{s.t.} \quad \sum_{s=k(\boldsymbol{\tau})}^{T} a_s (\overline{h}_s - h_s(\boldsymbol{\tau}, \boldsymbol{\tau})) \quad \leq \quad \delta \quad \forall \overline{\mathbf{h}} \in H_0. \tag{10}$$

Portfolio $\boldsymbol{\tau}$ is FSD optimal if and only if $\delta^*(H(\boldsymbol{\tau})) = 0$. If $\delta^*(H_0) > 0$ for some $H_0 \subseteq H(\boldsymbol{\tau})$ then $\boldsymbol{\tau}$ is FSD non-optimal.

The idea of this result is to find a representative utility function for which $\boldsymbol{\tau}$ maximizes expected utility. Note that $\xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau})) = \delta^*/T$. Since $a \in A(\boldsymbol{\tau})$ and $\mathbf{h} \in \{0, \ldots, T\}^T$ for all $\mathbf{h} \in H(\boldsymbol{\tau})$, using Corollary 1(i), we have $0 \leq \xi(\boldsymbol{\tau}, \Lambda(\boldsymbol{\tau})) \leq 1$.

Among other things, the theorem implies the following about the relationship between the efficiency concepts of optimality and admissibility.

Corollary 2:

If $(T \leq 4)$ then FSD optimality is equivalent to FSD admissibility.

Proof:

Without loss of generality, let $T = 4$ and let $\boldsymbol{\tau}$ be FSD admissible. Consider all possible $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ which are not dominated by each other[6]: $\mathbf{h}^1(\boldsymbol{\lambda}, \boldsymbol{\tau}) = (4, 2, 2, 2)$, $\mathbf{h}^2(\boldsymbol{\lambda}, \boldsymbol{\tau}) = (4, 3, 3, 0)$, $\mathbf{h}^3(\boldsymbol{\lambda}, \boldsymbol{\tau}) = (4, 4, 2, 0)$ and $\mathbf{h}^4(\boldsymbol{\lambda}, \boldsymbol{\tau}) = (4, 4, 1, 1)$. Entering these candidates in the linear programming test in Theorem 2, we can see that $\boldsymbol{\tau}$ is the optimal portfolio for a representative utility function with $a_2 = a_3 = a_4 = 1/3$, and hence $\boldsymbol{\tau}$ is FSD optimal.

[6] A dominated $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ can not change the solution of (9)-(10).

The numerical example in the next section shows that the two efficiency concepts diverge for $T \geq 5$.

A remaining problem is identifying the elements of the set $H(\boldsymbol{\tau})$. We may adopt several strategies for this task. The appendix provides a mixed integer linear programming (MILP) algorithm that identifies a set of candidate vectors $\widetilde{H}(\boldsymbol{\tau}) \supseteq H(\boldsymbol{\tau})$, and checks if $\mathbf{h} \in H(\boldsymbol{\tau})$ for every candidate $\mathbf{h} \in \widetilde{H}(\boldsymbol{\tau})$. A drawback of this approach is that the number of candidates increases exponentially with the number of scenarios (T). Hence, for large numbers of scenarios, this strategy may become computationally prohibitive and some sort of approximation may then be required.

For example, we may form a sample $H_s(\boldsymbol{\tau})$ of elements $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ by using a sample $\Lambda_s \in \Lambda(\boldsymbol{\tau})$ and constructing the associated values for $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$. The test procedure is then applied to the sample $H_s(\boldsymbol{\tau})$ instead of the complete set $H(\boldsymbol{\tau})$. [7] According to Corollary 1(ii), this will lead to a necessary condition for FSD optimality. There exist various techniques for performing the sampling task, including a regular grid, Monte Carlo methods or Quasi-Monte Carlo methods; see, for example, Jackel (2002) and Glasserman (2004).

While the MILP algorithm starts from a large set of candidate vectors and checks feasibility for every candidate, sampling from the portfolio space avoids searching over infeasible candidates. Of course, the limitation of this strategy is that the critical sample size needed to obtain an accurate approximation increases exponentially as the number of individual choice alternatives (N) increases. Still, this approach can yield an accurate approximation in an efficient manner if N is low. This is true especially when the correlation between the individual choice alternatives is high and hence small changes in the portfolio weights do not lead to large changes in the values of $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$.

An alternative approach is to enrich the Bawa *et al.* test by including the same sample of diversified portfolios Λ_s as additional choice alternatives. This will lead to a more powerful necessary condition for FSD optimality than considering the undiversified choice alternatives only. However, using the sample Λ_s in our test generally leads to a more favorouble trade-off between computation time and numerical accuracy.

Specifically, if we apply the Bawa *et al.* test to a grid with step size s, the relevant linear program has $M \cdot T$ columns and M rows, see (Bawa *et al.* (1985), Section IC, LP problem at the bottom of p. 423), or dimensions $M \cdot T \times M$, while the dimensions of our linear program (9)-(10) are $T \times M$, where

$$M = \prod_{i=1}^{N-1} (1 + \frac{1}{si})$$

[7] Since every $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ is known to be feasible, we can skip Step 2-5 of the algorithm and take only Step 1 and Step 6. Step 1 in this case boils down to performing pairwise dominance tests between every sampled portfolio and the evaluated portfolio. The computational burden of the step can be ignored.

is the number of portfolios from the grid. For example, if we use $T = 120$ time-series observations, $N = 10$ base assets and grid step size $s = 0.1$, the Bawa *et al.* test has dimensions $1.11 \cdot 10^7 \times 9.24 \cdot 10^4$, while our program has dimensions $120 \times 9.24 \cdot 10^4$.

V Numerical example

A numerical example can illustrate our test and the difference with the Bawa *et al.* test and Kuosmanen test. We focus on an example with five scenarios ($T = 5$), because FSD optimality is equivalent to FSD admissibility for ($T \leq 4$) (see Corollary 2).

Table 1 shows the returns to three choice alternatives (X_1, X_2, X_3) and the tested portfolio $Z = 0.16X_1 + 0.21X_2 + 0.63X_3$ in the five scenarios $(1, 2, 3, 4, 5)$.

[Insert Table 1 about here]

One can immediately see that no individual choice alternative (X_1, X_2 and X_3) FSD dominates Z; no other alternative involves a 100% chance of a return above -2% and a 20% chance of a return above 7%. However, this does not mean that Z is an optimal portfolio. Therefore, it is interesting to employ the three efficiency tests.

To implement the Kuosmanen test, we need to solve the following LP problem for each of the $5! = 120$ permutations of Z, say $\mathbf{y}_j = (y_j^1, y_j^2, y_j^3, y_j^4, y_j^5)$, $j = 1, 2, \ldots, 120$, or an equivalent mixed integer linear problem:

$$\begin{aligned}
\Psi_j = \max_{\lambda_1, \lambda_2, \lambda_3} \quad & \frac{1}{5}\sum_{t=1}^{5}(\lambda_1 x_1^t + \lambda_2 x_2^t + \lambda_3 x_3^t - y_j^t) \\
\text{s.t.} \quad \lambda_1 x_1^t + \lambda_2 x_2^t + \lambda_3 x_3^t & \geq y_j^t \quad t = 1, 2, 3, 4, 5 \\
\lambda_1 + \lambda_2 + \lambda_3 & = 1 \\
\lambda_1, \lambda_2, \lambda_3 & \geq 0
\end{aligned}$$

We find $\Psi_j^* = 0$ for every $j = 1, 2, \ldots, 120$, and hence Z is in the FSD admissible set (not FSD dominated by any convex combination of X_1, X_2 and X_3).

To test FSD optimality according to Bawa *et al.*, we need to establish if some convex combination of the CDFs of X_1, X_2 and X_3 dominates the CDF of Z, see (Bawa *et al.*, (1985), p. 421, Eq. 5). Table 2 shows the CDFs of the three choice alternatives (Φ_{X_1}, Φ_{X_2}, Φ_{X_3}) and the CDF of Z (Φ_Z). Note that these CDFs need to be evaluated only at the observed return levels: $\{z_j\}_{j=1}^{19}$.

[Insert Table 2 about here]

To implement the test, we need to solve the following LP problem, see (Bawa *et al.* (1985), Section IC, LP problem at the bottom of p. 423):

$$\eta = \max_{\lambda_1,\lambda_2,\lambda_3} \sum_{j=1}^{19} (\Phi_Z(z_j) - \lambda_1 \Phi_{X_1}(z_j) - \lambda_2 \Phi_{X_2}(z_j) - \lambda_3 \Phi_{X_3}(z_j))$$

$$\begin{aligned} \text{s.t.} \quad \lambda_1 \Phi_{X_1}(z_j) + \lambda_2 \Phi_{X_2}(z_j) + \lambda_3 \Phi_{X_3}(z_j) &\leq \Phi_Z(z_j) \quad j = 1, \ldots, 19 \\ \lambda_1 + \lambda_2 + \lambda_3 &= 1 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0 \end{aligned}$$

Solving this problem, we find $\eta^* = 0$, and hence Z is classified as optimal; not every nonsatiable decision-maker will prefer X_1 or X_2 or X_3 to Z. Based on the positive outcomes of the two tests, we may be tempted to conclude that Z is the optimal portfolio for some increasing utility function. Perhaps surprisingly, this conclusion is wrong. The application of our MILP algorithm will demonstrate this. We will follow the steps outlines in the Appendix.

Since we have already tested FSD admissibility, we start with the second step of identifying the initial candidates for $H(\boldsymbol{\tau})$. For $j = 2, 3, 4, 5$, we solve (11), where $k(\boldsymbol{\tau}) = 2$, $T = 5$, $\underline{m} = -4$, $\overline{m} = 10$ and $X\boldsymbol{\tau} = Z$. (Recall that the constants $\underline{m}$ and $\overline{m}$ are the minimal and maximal possible returns, and $k(\boldsymbol{\tau})$ is the order of the second smallest return of $\boldsymbol{\tau}$.) Table 3 shows the optimal solutions for $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ and $\boldsymbol{\lambda}$. It follows that $\mathbf{h}^{max} = (5, 5, 4, 3, 2)$.

[Insert Table 3 about here]

In this example, we find $\Lambda_1 = \{(0.1483, 0.8517, 0), (0.1187, 0.8813, 0), (0.9266, 0.0734, 0)\}$, and $H_1 = \{(5, 5, 4, 2, 0), (5, 5, 3, 3, 0), (5, 3, 3, 2, 2)\}$ for the set of corresponding values of $\mathbf{h}^*$.

In the third step, we apply the stopping rules for the initial candidates. Since $\mathbf{h}(\boldsymbol{\tau}, \boldsymbol{\tau}) = (5, 4, 3, 2, 1)$, $h_t^{max} > h_t(\boldsymbol{\tau}, \boldsymbol{\tau})$ for all $t = k(\boldsymbol{\tau}), ..., T$, hence the sufficient condition of FSD optimality is not fulfilled. Since $\xi(\boldsymbol{\tau}, \Lambda_1) = 0$, the necessary condition of FSD optimality is also not fulfilled; there exists a decision-maker who prefers $\boldsymbol{\tau}$ to all portfolios in Λ_1.

Thus, we proceed with the fourth step of constructing and reducing the candidate set $\overline{H}$. Since $\mathbf{h}^{max} = (5, 5, 4, 3, 2)$, the candidate set consists of $6 * 6 * 5 * 4 * 3 = 2160$ elements. We exclude candidates for which a corresponding portfolios can not exist, that is, the members of the sets $\widetilde{H} = \widetilde{H}_1 \cup \widetilde{H}_2 \cup \widetilde{H}_3 \cup \widetilde{H}_4$. The remaining candidates are:

$$
\begin{aligned}
\mathbf{h}_c^1 &= (5,5,4,1,1)\\
\mathbf{h}_c^2 &= (5,5,2,2,2)\\
\mathbf{h}_c^3 &= (5,5,2,2,1)\\
\mathbf{h}_c^4 &= (5,5,2,1,1)\\
\mathbf{h}_c^5 &= (5,5,1,1,1)\\
\mathbf{h}_c^6 &= (5,4,4,1,1)\\
\mathbf{h}_c^7 &= (5,4,2,2,2)\\
\mathbf{h}_c^8 &= (5,3,3,3,1).
\end{aligned}
$$

Finally, we employ the last two steps of our algorithm. Step 5 tests feasibility of a remaining candidate using (12). If the candidate is infeasible then we choose the next one. If the candidate is feasible then we add it to H_1 and we recompute $\xi(\boldsymbol{\tau}, H_1)$. Let us start with $\mathbf{h}_c^1 = (5,5,4,1,1)$. This candidate is feasible as it corresponds to $\boldsymbol{\lambda} = (0.265, 0.735, 0)$. Adding this candidate, we consider $\Lambda_2 = \Lambda_1 \cup (0.265, 0.735, 0)$ and $H_2 = H_1 \cup (5,5,4,1,1)$. Applying Theorem 2, we solve the following linear problem:

$$
\begin{array}{lrrrrcl}
 & \multicolumn{6}{c}{\min \delta} \\
\text{s.t.} & a_2 & +a_3 & & -a_5 & \leq & \delta \\
 & a_2 & & +a_4 & -a_5 & \leq & \delta \\
 & -a_2 & & & +a_5 & \leq & \delta \\
 & a_2 & +a_3 & -a_4 & & \leq & \delta \\
 & a_2 & +a_3 & +a_4 & +a_5 & = & 1
\end{array}
$$

We find $\delta^* = 1/9$, $\xi(\boldsymbol{\tau}, \Lambda_2) = \delta^*/5 = 1/45 > 0$. This means that we can not find a representative utility function that rationalizes the evaluated portfolio. Thus, adding portfolio $(0.265, 0.735, 0)$ to Λ_1 suffices to demonstrate non-optimality in this case. Note that this portfolio does not dominate the evaluated portfolio, as the evaluated portfolio is FSD admissible. However, we do know that every well-behaved investor will prefer $(0.265, 0.735, 0)$ or an element of Λ_1 to the evaluated portfolio. Since the evaluated portfolio is classified as FSD non-optimal, the algorithm is complete. Thus, in this example, Z is classified as optimal according to Bawa *et al.* and Kuosmanen tests. Still, it can be demonstrated to be non-optimal for any increasing utility function.

We may repeat this exercise for more portfolios $\boldsymbol{\tau} \in \Lambda \cap \{0, 0.01, \ldots, 1\}^3$, that is, when using a grid with step size 0.01 for the portfolio weights. Figure 2 illustrates the comparison between FSD admissibility and FSD optimality.

[Insert Figure 2 about here]

The Kuosmanen test recognizes that many diversified portfolios are FSD dominated by other diversified portfolio, most notably those that assign a high weight to X_3. In this example, only 22 % of the considered portfolios are FSD admissible (the union of the grey and black dots). The FSD optimal set is even smaller than the admissible set. The set of grey dots, including Z, is now excluded, leaving only the black dots. The reduction in the efficient set to 16 % of all considered portfolios (a 26 % reduction) is possible because the optimality test acknowledges that a choice alternative may not be optimal for all investors even if no single other choice is preferred by all. Note that the efficient regions are not convex, witness for example the small isolated optimal area near $\boldsymbol{\lambda} = (0, 0.7, 0.3)$.

A similar analysis can be done for FSD optimality according to Bawa *et al.* (1985). Figure 3 shows that 93 % of all portfolios is classified as optimal. Only 17 % of these portfolios are FSD optimal. The optimal set is substantially larger than ours, because the Bawa optimality test does not account for full diversification.

[Insert Figure 3 about here]

As discussed in Section 3, we can increase the power of the Bawa *et al.* test by adding a grid of diversified portfolios to the individual choice alternatives. Of course, this approach will still yield only a necessary condition, because it is computationally impossible to include all infinitely many relevant portfolios. In addition, using the same grid of diversified portfolios in our test will lead to a smaller linear program. Figure 4 shows the set of portfolios which are not classified as FSD non-optimal using the enriched Bawa *et al.* test and our test using the same grid step size.

[Insert Figure 4 about here]

There are only small differences in the power of the two tests for $s = 0.1$. However, our test is roughly 120 times faster than the enriched Bawa *et al.* test. For $s = 0.01$, our test is very powerful: 97% of non-optimal portfolios are correctly classified as non-optimal. Unfortunately, we were unable to implement the enriched Bawa *et al.* test for this step size due to the excessive computation load. The differences in computation load will be even larger for real-life applications with higher dimensions.

VI Empirical application

To further illustrate our test, we apply it to US stock market data in order to analyze FSD optimality of the market portfolio relative to portfolios formed on market capitalization of equity (size) and book-to-market equity ratio (B/M). This test seems relevant for asset pricing theory, because all single-period, portfolio-oriented, representative-investor models of capital market equilibrium predict that the market portfolio is optimal for a representative investor with well-behaved preferences.

The investment universe of stocks is proxied by the well-known six value-weighted Fama and French portfolios constructed as the intersection of two groups formed on size (small caps and large caps) and three groups formed on B/M (growth stock, neutral stocks and value stocks). We proxy the market portfolio by the CRSP all-share index, a value weighted average of common stocks listed on NYSE, AMEX, and NADAQ, and the riskless asset by the one-year US government bond index from Ibbotson Associates. We consider yearly (January-December) excess returns from 1963 to 2002 (40 annual observations).[8,9] Excess returns are computed by subtracting the riskless rate from the nominal returns, that is, the riskless asset always has a return of zero.

Table 4 shows some descriptive statistics for our data set. Particularly puzzling is the value premium in the small cap segment. The small value stocks earned an average annual excess return of 13.86 percent, 8.55 percent in excess of the 5.31 percent for small growth stocks. It seems difficult to explain away this premium with risk because the small growth stocks actually have a higher standard deviation than the small value stocks. Indeed, the market portfolio is SSD inefficient, as shown before by Post (2003). This means that in the face of attractive premiums from investing in small caps stocks and value stocks, investing in the market portfolio seems not optimal for any risk averse investor.

[Insert Table 4 about here]

Still, the market portfolio may be FSD optimal, for example, it may be optimal for investors who are risk seeking for losses and risk averse for gains. Our first step to analysing FSD optimality is to apply the Bawa *et al.* test. This test classifies the market portfolio as optimal, meaning that some investors prefer the market portfolio to all of the 7 benchmark portfolios (six Fama and French, and the riskless asset). However, as discussed before, the test does not account for diversification between the seven portfolios. To analyze the effect of diversification, we can enrich the Bawa *et*

[8] As discussed in Benartzi and Thaler (1995, p.83), one year is a plausible choice for the investor's evaluation period, because "individual investors file taxes annually, receive their most comprehensive reports from their brokers, mutual funds, and retirement accounts once a year, and institutional investors also take the annual reports most seriously." Excess returns are computed by subtracting the riskless rate from the nominal returns.

[9] There are two reasons for starting in 1963 and omitting the pre-1963 data. First, prior to 1963, the Compustat database is affected by survivorship bias caused by the backfilling procedure excluding delisted firms, which typically are less successful (Kothari, Shanken and Sloan (1995)). Further, from June 1962, AMEX-listed stocks are added to the CRSP database, which includes only NYSE-listed stocks before this month. Since AMEX stocks generally are smaller than NYSE stocks, the relative number of small caps in the analysis increases from June 1962. Since the value effect is most pronounced in the small-cap segment, the post-1962 data set is most challenging.

al. test by adding diversified portfolios or apply the Kuosmanen test. Using the grid $\Lambda_g = \Lambda(\boldsymbol{\tau}) \cap \{0, 0.1, \ldots, 1\}^7$ the enriched Bawa *et al.* test already leads to linear program with more than 320,000 constrains and 8,000 variables. We therefore apply the Kuosmanen test, which involves solving a mixed-integer program with 1,600 integer variables. Interestingly, this test classifies the market portfolio as FSD inadmissible and identifies the dominating portfolio shown in Figure 5.

[Insert Figure 5 about here]

Since FSD inadmissibility implies FSD non-optimality there is no need to apply our test in this case. Still, it is useful to apply our test for the purpose of illustration and comparison of the complexity of these three tests.

Since the number of choice alternatives (7) is small in comparison to the number of scenarios (40), we apply the method of sampling portfolios using the grid: $\Lambda_g = \Lambda(\boldsymbol{\tau}) \cap \{0, 0.1, \ldots, 1\}^7$. The associating vectors $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ are collected in H_g and H_g is used to proxy for $H(\boldsymbol{\tau})$ in the linear programming problem (9)-(10). This linear program has only 8,000 constrains and 40 variables. Therefore our test is much more faster than both the Kuosmanen test and enriched Bawa *et al.* test for the same grid. Interestingly, the non-optimality measure is strictly positive; $\xi(\boldsymbol{\tau}, \Lambda_g) = 0.00275$. According to Corollary 1(ii), this implies that the market portfolio is not optimal for any increasing utility function.

Table 5 illustrates the non-optimality classification. It shows 9 combinations of the 7 benchmark portfolios. For the vectors $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$ associated with these combinations, the restrictions in (9)-(10) are binding. This means that the value of the non-optimality measure critically depends on these vectors. By contrast, the other vectors can be excluded without affecting the non-optimality measure. None of these 9 combinations FSD dominates the evaluated portfolio. Still, for every increasing utility function, at least one of these combinations is better than the market portfolio. Not surprisingly, each of these portfolios assigns a substantial weight to small cap stocks and/or value stocks.

[Insert Table 5 about here]

The above analysis focuses on sample optimality. It is desirable to account for sampling error and establish the statistical confidence we have in population optimality. For mean-variance efficiency tests, the sampling distribution is well-known, see, for example Gibbons, Ross and Shanken (1989). The sampling distribution for SD tests is more difficult to derive, because the shape of the population return distribution is not restricted. We therefore resort to the bootstrap method, a well-established tool to analyze the sensitivity of empirical estimators to sampling variation in situation where the sampling distribution is difficult to obtain analytically.

Under the assumption of serially IID returns, the empirical return distribution is a consistent estimator of the population return distribution, and bootstrapping samples can simply be obtained by randomly sampling with replacement from the empirical return distribution. Nelson and Pope (1991) demonstrated in a convincing way that this approach can quantify the sensitivity of the empirical return distribution to sampling variation, and that SD analysis based on the bootstrapped return distribution is more powerful that analysis based on the original empirical return distribution. We implement this method by generating 10,000 random pseudo-samples and apply our tests for each pseudo-sample. We do not apply the enriched Bawa *et al.* test or the Kuosmanen test, because of the associated computational burden. Rather, we apply our LP necessary test (9)-(10) using the 9 combinations from Table 5. In 97.9 % of the pseudo-samples, the market portfolio did not pass this necessary test. Then, for the remaining 2.1% of the pseudo-samples, we apply our LP necessary test (9)-(10) using the grid $\Lambda_g = \Lambda(\boldsymbol{\tau}) \cap \{0, 0.1, \ldots, 1\}^7$. In 0.8% of the pseudo-samples, the market portfolio failed this necessary test. For the remaining 1.3% of the pseudo-samples, we applied our necessary and sufficient test. The market portfolio was classified as FSD optimal in all of these pseudo-samples. Thus, bootstrap p-value is 1.3% and the market portfolio can be classified as significantly FSD non-optimal with 98.7% confidence.

The classification of the market portfolio as FSD non-optimal reinforces Post's (2003) finding that the market portfolio is SSD inefficient. This finding is potentially important for asset pricing theory. All single-period, portfolio-oriented, representative-investor models predict FSD optimality. FSD non-optimality would contradict all these models and may call for multi-period models, consumption-oriented models or heterogeneous-investor models. However, we stress that this application serves only to illustrate our non-optimality test. Among other things, the choice of the benchmark portfolios and market portfolio, investment horizon and sample period requires more analysis than is possible here.

VII Conclusions

We have developed a test for "FSD efficiency" of a given portfolio that is more powerful than currently available. In contrast to Bawa *et al.* (1985), our test compares the evaluated portfolio not only with the finite set of individual choice alternatives, but also with all portfolios formed by combining the individual alternatives. In contrast to Kuosmanen (2004), our efficiency test is based on the criterion of FSD optimality rather than the weaker criterion of FSD admissibility.

The test can be performed by solving a simple linear programming problem. However, the input to the linear programming problem may require an initial phase of mixed integer linear programming (MILP). For large numbers of scenarios, this strategy may become computationally prohibitive and we may have to resort to an approximation based on sampling portfolios from the portfolio possibilities set. This subsampling approach improves the trade-off between computational complexity and numerical accuracy compared with enriching the Bawa *et al.* test with diversified choice alternatives.

Using our new test, we showed that the US stock market portfolio is significantly FSD non-optimal relative to benchmark portfolios formed on market capitalization and book-to-market equity ratio; no nonsatiable investor would hold the market portfolio in the face of the small cap premium and the value stock premium. FSD non-optimality would contradict all single-period, portfolio-oriented, representative-investor models of capital market equilibrium and would call for multi-period models, consumption-oriented models or heterogeneous-investor models. The focus of our study is however on methodology and a rejection of market portfolio optimality requires a more rigorous empirical analysis than is possible in this study.

Appendix

This appendix provides a MILP algorithm for identifying the elements of $H(\boldsymbol{\tau})$ and suggests some stopping rules for testing FSD optimality.

STEP 1: *Perform a FSD admissibility test*

As an initial stopping rule, test FSD admissibility of $\boldsymbol{\tau}$, for example using the MILP test of Kuosmanen (2004). If $\boldsymbol{\tau}$ is FSD inadmissible then stop the algorithm; $\boldsymbol{\tau}$ is FSD non-optimal.

STEP 2: *Identify initial candidates for $H(\boldsymbol{\tau})$*

For all $j = k(\boldsymbol{\tau}), ..., T$ solve the following MILP problem:

$$
\text{(11)} \qquad \max \quad h_j + \frac{1}{T^2}\sum_{t=k(\boldsymbol{\tau})}^{T} h_t
$$

$$
\begin{aligned}
\text{s.t.} \quad (v_{s,t}-1)(\overline{m}-\underline{m}) &\leq \mathbf{x}^s\boldsymbol{\lambda} - (X\boldsymbol{\tau})^{[t]} \leq v_{s,t}(\overline{m}-\underline{m}) && s = 1,\ldots,T;\ t = k(\boldsymbol{\tau}),\ldots,T \\
h_t &= \sum_{s=1}^{T} v_{s,t} && t = k(\boldsymbol{\tau}),\ldots,T \\
v_{s,t} &\in \{0,1\} && s = 1,\ldots,T;\ t = k(\boldsymbol{\tau}),\ldots,T \\
\boldsymbol{\lambda} &\in \Lambda(\boldsymbol{\tau})
\end{aligned}
$$

The problem is solved only for $j >= k(\boldsymbol{\tau})$; solving it for $j < k(\boldsymbol{\tau})$ will identify no new candidates, because the optimal solutions of (11) for any $j < k(\boldsymbol{\tau})$ is equal to that for $j = k(\boldsymbol{\tau})$.

Use $(h_t^{*j}, \lambda_t^{*j}, v_{s,t}^{*j})$ for the optimal solution of this problem. Let $\Lambda_1 \in \Lambda(\boldsymbol{\tau})$ be a set of pairwise different $\boldsymbol{\lambda}^{*j}$ (all redundancies are removed). Set

$$
\begin{aligned}
h_t^{max} &= \max_j h_t^{*j} \\
H_1 &= \{\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau}) : \quad \boldsymbol{\lambda} \in \Lambda_1\}
\end{aligned}
$$

STEP 3: *Stopping rules*

Consider $\mathbf{h}(\boldsymbol{\tau}, \boldsymbol{\tau})$ as defined by (6)-(7). If there exists $t \in \{k(\boldsymbol{\tau}), \dots, T\}$ such that $h_t^{max} \leq h_t(\boldsymbol{\tau}, \boldsymbol{\tau})$ then stop the algorithm; $\boldsymbol{\tau}$ is FSD optimal. Otherwise, solve problem (9)-(10) for $H_0 = H_1$. If $\delta^*(H_1) > 0$ then stop the algorithm; $\boldsymbol{\tau}$ is FSD non-optimal.

STEP 4: *Construct and reduce the candidate set* $\overline{H}$

Let $\overline{H}_t = \{0, 1, \dots, h_t^{max}\}$. Use $\overline{H}$ for the cartesian product $\overline{H} = \bigotimes_{k(\boldsymbol{\tau})}^{T} \overline{H}_t$. Clearly $H(\boldsymbol{\tau}) \subseteq \overline{H}$, and hence $\overline{H}$ is a candidate set. Exclude the candidates $\widetilde{H} = \widetilde{H}_1 \cup \widetilde{H}_2 \cup \widetilde{H}_3 \cup \widetilde{H}_4$, where

$$
\begin{aligned}
\widetilde{H}_1 &= \left\{\mathbf{h} \in \overline{H} | h_{t_1} < h_{t_2} \text{ for some } t_1 < t_2\right\} \\
\widetilde{H}_2 &= \left\{\mathbf{h} \in \overline{H} | h_t \geq h_t(\boldsymbol{\tau}, \boldsymbol{\tau}) \;\; \forall t \in \{k(\boldsymbol{\tau}), \dots, T\}\right\} \\
\widetilde{H}_3 &= \Big\{\mathbf{h} \in \overline{H} | \exists \widehat{\mathbf{h}} \in H_1 : \;\; h_t \geq \widehat{h}_t \;\; \forall t \in \{k(\boldsymbol{\tau}), \dots, T\} \text{ with at least one} \\
&\quad \text{strict inequality}\}
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{H}_4 = & \left\{ \mathbf{h} \in \overline{H} | h_t \leq \xi h_t(\boldsymbol{\tau}, \boldsymbol{\tau}) + (1 - \xi) \sum_{j = k(\boldsymbol{\tau})}^{T} \eta_j h_t^{*j}, \;\; \forall t \in \{k(\boldsymbol{\tau}), \dots, T\}, \right. \\
& \left. \forall \mathbf{h}^{*j} \in H_1, \;\; 0 \leq \xi \leq 1, \;\; \sum_{j = k(\boldsymbol{\tau})}^{T} \eta_j = 1, \;\; \eta_j \geq 0, \;\; \forall j \in \{k(\boldsymbol{\tau}), \dots, T\} \right\}.
\end{aligned}
$$

The elements of $\widetilde{H}$ are not feasible, that is, there exist no corresponding portfolios. The elements of $\widetilde{H}_1$ contradict the definition of vector $\mathbf{h}(\boldsymbol{\lambda}, \boldsymbol{\tau})$, see (6)-(7). In step 1, we have found that $\boldsymbol{\tau}$ is FSD admissible. Feasibility of an element of $\widetilde{H}_2$ implies FSD inadmissibility of $\boldsymbol{\tau}$. Every element of $\widetilde{H}_3$ gives a strictly higher value of the objective function in (11) for at least one initial candidate. Thus it can not be a feasible candidate.

Adding the elements of $\widetilde{H}_4$ to H_1 does not affect the solution of (9)-(10).

Set $p = 1$.

STEP 5: *Check feasibility of the remaining candidates*

If $\overline{H} \setminus \widetilde{H}$ is empty, that is, all possible $\mathbf{h} \in \overline{H}$ have been considered, then stop the algorithm; portfolio $\boldsymbol{\tau}$ is FSD optimal. Otherwise, choose $\mathbf{h} \in \overline{H} \setminus \widetilde{H}$ and add it to $\widetilde{H}$. Let $p = p+1$, $H_p = H_{p-1} \cup \mathbf{h}$ and go to the next step if there exists a feasible solution of the system:

$$
\text{(12)} \quad \text{s.t.} \quad \begin{array}{rcll}
(v_{s,t}-1)(\overline{m}-\underline{m}) & \leq & \mathbf{x}^s\boldsymbol{\lambda} - (X\boldsymbol{\tau})^{[t]} \leq v_{s,t}(\overline{m}-\underline{m}) & s = 1,\dots,T; \\
 & & & t = t_1,\dots,T \\
h_t & = & \sum_{s=1}^{T} v_{s,t} & t = t_1,\dots,T \\
v_{s,t} & \in & \{0,1\} & s = 1,\dots,T; \\
 & & & t = t_1,\dots,T \\
\boldsymbol{\lambda} & \in & \Lambda(\boldsymbol{\tau}) &
\end{array}
$$

Otherwise, repeat this step.

STEP 6: *Test optimality using the feasible candidates*

Solve problem (9)-(10) for $H_0 = H_p$. If $\delta^*(H_p) > 0$ then stop the algorithm; $\boldsymbol{\tau}$ is FSD non-optimal. Otherwise, go to Step 5.

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Table 1: *Example showing that the Bawa et al. test and the Kuosmanen test do not give a sufficient condition for FSD optimality.*
The table shows the returns in five scenarios to three choice individual alternatives (X_1, X_2 and X_3) and the tested portfolio $Z = 0.16X_1 + 0.21X_2 + 0.63X_3$. No convex combination of X_1, X_2 and X_3 FSD dominates Z and hence Z is FSD admissible.

t	X_1	X_2	X_3	Z
1	-1	6	-4	-1.42
2	-2	5.90	2	2.18
3	3.50	2.20	3	2.91
4	8.70	2	5	4.96
5	10	7	7.50	7.80

Table 2: *Example showing that the Bawa et al. test and Kuosmanen test do not give a sufficient condition for FSD optimality–continued.*
The table shows the CDFs of the three individual choice alternatives (X_1, X_2, X_3) and the tested portfolio Z for all observed return levels. No convex combination of Φ_{X_1}, Φ_{X_2}, Φ_{X_3} dominates Φ_Z and hence Z is classified as optimal.

j	z_j	Φ_{X_1}	Φ_{X_2}	Φ_{X_3}	Φ_Z
1	-4	0	0	1/5	0
2	-2	1/5	0	1/5	0
3	-1.42	1/5	0	1/5	1/5
4	-1	2/5	0	1/5	1/5
5	2	2/5	1/5	2/5	1/5
6	2.18	2/5	1/5	2/5	2/5
7	2.2	2/5	2/5	2/5	2/5
8	2.91	2/5	2/5	2/5	3/5
9	3	2/5	2/5	3/5	3/5
10	3.5	3/5	2/5	3/5	3/5
11	4.962	3/5	2/5	3/5	4/5
12	5	3/5	2/5	4/5	4/5
13	5.9	3/5	3/5	4/5	4/5
14	6	3/5	4/5	4/5	4/5
15	7	3/5	1	4/5	4/5
16	7.5	3/5	1	1	4/5
17	7.795	3/5	1	1	1
18	8.7	4/5	1	1	1
19	10	1	1	1	1

Table 3: *Initial candidates.*
The table presents the initial candidates H_1 and the associated $\Lambda_1(\boldsymbol{\tau})$ obtained in Step 2 of our algorithm.

j	h_1^*	h_2^*	h_3^*	h_4^*	h_5^*	λ_1^*	λ_2^*	λ_3^*
2	5	5	4	2	0	0.1483	0.8517	0
3	5	5	4	2	0	0.1483	0.8517	0
4	5	5	3	3	0	0.1187	0.8813	0
5	5	3	3	2	2	0.9266	0.0734	0

Table 4: *Descriptive statistics.*
The table shows descriptive statistics for the annual (January-December) excess returns of the six Fama and French stock portfolios formed on market capitalization of equity and book-to-market equity ratio (SG=small growth, SN=small neutral, SV=small value, BG=big growth, BN=big neutral and BV=big value), and the CRSP all-equity index (CRSP). Excess returns are computed by subtracting the return to the one-year US government bond from the nominal returns. The sample period is from 1963 to 2002 (40 annual observations). Equity data are from Kenneth French and bond data are from Ibbotson Associates.

	Mean	St.dev.	Skew.	Kurt.	Min.	Max.
SG	5.309	28.520	0.323	0.175	-49.28	83.68
SN	11.301	22.728	-0.308	0.062	-37.38	65.48
SV	13.861	23.158	-0.373	-0.222	-33.86	61.14
BG	5.303	18.820	-0.317	-0.537	-40.49	34.67
BN	6.340	16.120	-0.241	-0.090	-34.13	34.73
BV	8.946	17.723	-0.690	-0.026	-34.24	40.34
CRSP	5.536	17.191	-0.602	-0.404	-39.19	31.89

Table 5: *Nine combinations showing FSD non-optimality of the market portfolio.*
The table shows the portfolio weights of 9 combinations of the six Fama and French stock portfolios formed on size and B/M (SG=small growth, SN=small neutral, SV=small value, BG=big growth, BN=big neutral and BV=big value), and the riskless alternative (RL). For every increasing utility function, at least one of these nine combinations is preferred to the market portfolio, and hence the market portfolio is FSD non-optimal.

Combination	SG	SN	SV	BG	BN	BV	RL
1	0	0	0.1	0.3	0.1	0.4	0.1
2	0	0	0.3	0.2	0	0.2	0.3
3	0	0	0.4	0	0	0.3	0.3
4	0	0	0.4	0	0.1	0.3	0.2
5	0	0	0.6	0	0.1	0	0.3
6	0	0	0.6	0.2	0	0.1	0.1
7	0	0.1	0.5	0.1	0.1	0.1	0.1
8	0	0.1	0.9	0	0	0	0
9	0	0.2	0.8	0	0	0	0

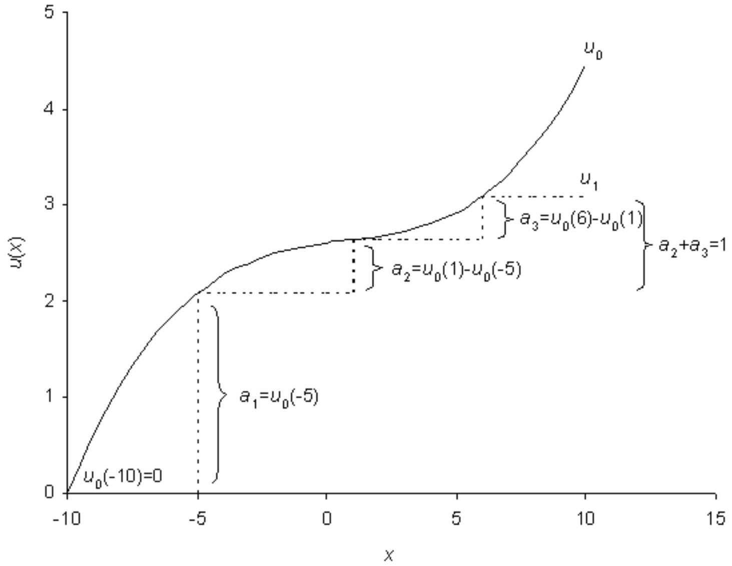

Figure 1: *Representative utility function.*
The figure shows the original utility function u_0 and the associated representative utility function u_1.

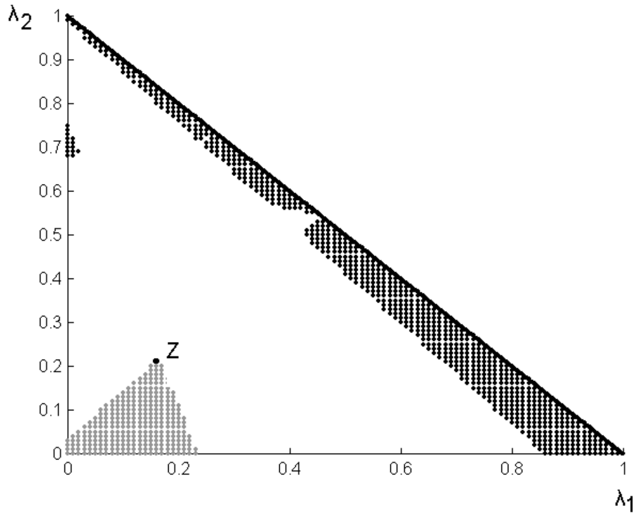

Figure 2: *Admissibility and optimality.*
The figure shows the efficiency classification according to the FSD admissibility test and our FSD optimality test. We applied these tests to all portfolios $\boldsymbol{\tau} \in \Lambda \cap \{0, 0.01, \ldots, 1\}^3$, that is, when using a grid with step size 0.01 for the portfolio weights. Our optimal set is represented by the black dots. The admissible set is the union of the black dots and the grey dots.

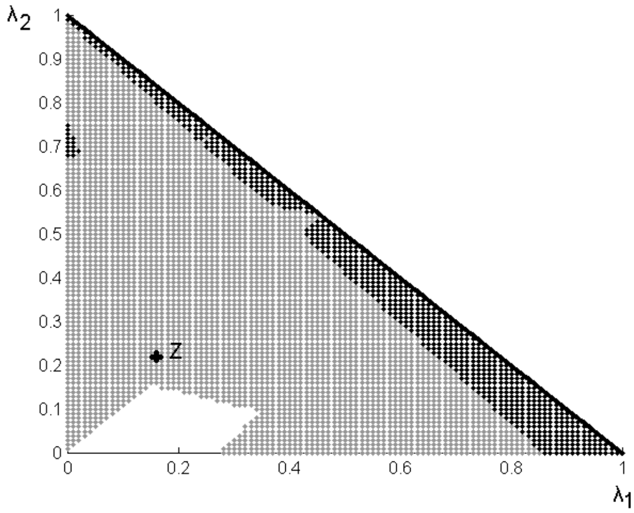

Figure 3: *Bawa et al. optimality and FSD optimality.*
This figure shows the optimality classification according to the Bawa *et al.* test and our test for FSD optimality. Our optimal set is represented by the black dots. The Bawa *et al.* optimal set is the union of the black dots and the grey dots.

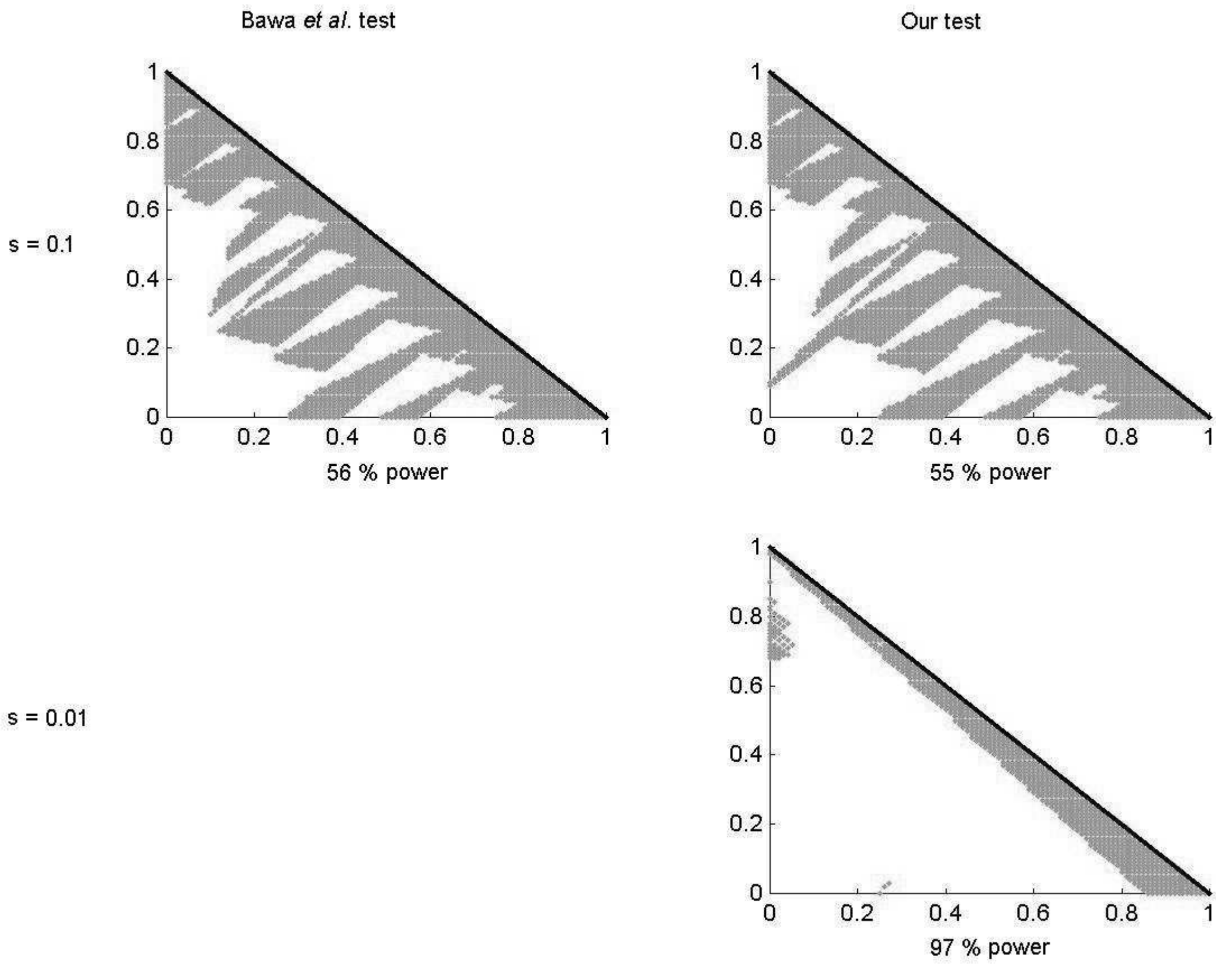

Figure 4: *Subsampling approach.*
The figure shows the outcomes of the Bawa *et al.* test and our test when applied to a grid of portfolios with step size s=0.1 or s=0.01. The grey dots are portfolios that passed the necessary test; the other portfolios failed the test and are classified as FSD non-optimal. The percentages of FSD non-optimal portfolios that are detected using the necessary tests are given below every graph.

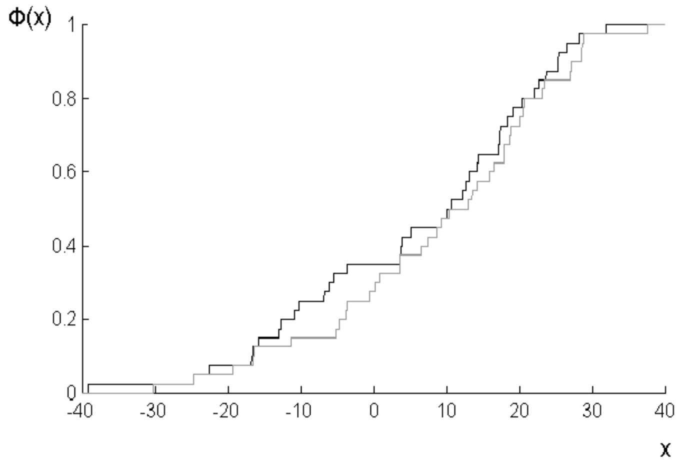

Figure 5: *Pairwise FSD dominance.*
This figure shows the CDF of the stock market portfolio (black line) and the dominating portfolio (grey line): $\boldsymbol{\lambda}_d = (0, 0.04, 0.43, 0.37, 0.04, 0, 0.13)$. Since the dominating portfolio is preferred by all investors, the market portfolio is FSD inadmissible.