Testing Portfolio Efficiency with Non-Traded Assets:

Taking into Account Labor Income, Housing and Liabilities

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JEL Classification: G11, G12

Keywords: Portfolio efficiency, Multivariate test, Asset pricing, Non-traded assets

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Abstract

This study extends the classical Gibbons, Ross and Shanken (1989) test for mean-variance efficiency of a given portfolio to include linear equality restrictions on the weights of a subset of restricted assets. The restricted assets can be thought of as illiquid or non-traded. This includes the relevant applications of a portfolio efficiency test while taking into non-traded labor income, housing and pension liabilities. Assuming a conditional normal distribution for the asset returns, we show that the test statistic follows an F-distribution in small samples.

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1. Introduction

Tests for mean-variance efficiency of a given portfolio are useful tools for portfolio management applications and empirical asset pricing research. Early efficiency tests such as the classical mean-variance efficiency tests of Jobson and Korkie (1980, 1982), Gibbons (1982), Kandel (1984), MacKinlay (1987), Shanken (1985, 1986), and Gibbons, Ross and Shanken (GRS; 1989) focus on the case where the portfolio weights are unrestricted. In this paper we consider a setting where the trading of a subset of assets is restricted by linear constraints. The restricted subset of assets can be thought of as illiquid, or when the portfolio weights are fixed at given values, as non-traded. Applications include tests of portfolio efficiency for investors with a substantial investment in housing, labor income, or non-traded liabilities.

An example of a relevant application is testing household portfolio efficiency while taking into account an illiquid investment in housing that cannot be adjusted in the short-term. Flavin and Yamashita (2002), Cocco (2004) and Hu (2005) show that a substantial investment in housing – typical for most individuals – can crowd stocks out of the investor's portfolio. Pelizzon and Weber (2003) test the efficiency of more than 5000 Italian household portfolios under the assumption that the individual's investment in housing is fixed. The results show that the constraint on the housing investment plays an important role in determining whether the portfolios are efficient. Our test is not only applicable in the growing field of household finance (for an overview, see Campbell, 2006), but also useful for institutional investors with non-traded liabilities, such as the liabilities arising from defined benefits pension schemes (see, e.g., Berkelaar and Kouwenberg, 2003).

Our analysis starts from the optimality conditions for mean-variance efficiency of a given portfolio under constraints. We formulate the null hypothesis of efficiency and propose a test statistic for measuring deviations from the null. Under the assumption of a normal distribution for the excess asset returns, we prove that the test statistic follows an F-distribution. The unrestricted classical GRS test is a special case within our framework. Apart from generalizing the GRS test, the contribution of the paper to the literature is that the test statistic is easily computed and suited for small samples, whereas available tests for efficiency under restrictions typically rely on approximations, large sample theory or computer simulation of the posterior distribution.

This paper aims to enrich the set of methods for testing mean-variance efficiency under constraints available in the literature. Wang (1998) extends the Bayesian approach for examining portfolio efficiency of Kandel *et al.* (1995) to include general restrictions on the portfolio weights. Similar to our paper, Wang (1998) assumes that asset returns are normally distributed. The posterior distribution of the efficiency measure is computed numerically with simulations. An advantage of the numerical approach is that the test can handle many different types of constraints. Further, Wang (1998) uses direct measures of the degree of portfolio efficiency, such as the maximum improvement in mean return given the variance of the evaluated portfolio. On the other hand, simulations can be time-consuming and some researchers might prefer the classical approach of hypothesis testing over the Bayesian approach (which uses posterior odds ratios, instead of p-values). We will not enter the debate about the relative merits of classical and Bayesian statistics here. Rather, our purpose is to extend the classical approach to testing mean-variance efficiency with a test that applies under restrictions on the portfolio weights.

Basak, Jagannathan and Sun (2002) develop a direct test for portfolio efficiency subject to short sale constraints. Similar to Wang (1998), Basak *et al.* (2002) measure the maximum improvement in variance that can be achieved by forming a portfolio of the primitive assets with the same mean as the benchmark portfolio. Basak *et al.* (2002) test whether the potential improvement is significantly greater than zero using a classical statistical approach, complementing the Bayesian approach followed by Wang (1998). Basak *et al.* (2002) prove that the sampling distribution of the estimated efficiency measure converges to a normal distribution as the number of observations goes to infinity. In order to derive this asymptotic result the paper applies a linear approximation method. Basak *et al.* (2002, p. 1213) report that the estimated efficiency measure is a non-linear function of the data in applications with short sale constraints and the linear approximation method might therefore introduce large errors.

Gouriéroux and Jouneau (1999) develop a mean-variance efficiency test for an investment setting where the portfolio weights of a subset of assets are fixed at given weights. Under the assumption of a multivariate normal asset return distribution, the test statistic proposed by Gouriéroux and Jouneau (1999) follows a chi-square distribution asymptotically. The test includes the unrestricted mean-variance test of Jobson and Korkie (1980, 1982) as a special case. Our paper complements the work of Gouriéroux and Jouneau (1999) by generalizing the classical test of Gibbons, Ross and Shanken (1989) to an investment setting with a subset of illiquid or non-traded assets. An advantage of our approach is that we find the exact small sample distribution of the test statistic (F-distribution). Further, our investment setting is slightly more general, as it includes linear restrictions on the entire subset of restricted assets. A relevant example of such a constraint is a binding limit on foreign investment.

Following GRS and others, we assume that the asset returns follow a joint normal distribution. As shown in Affleck-Graves and McDonald (1989), monthly US stock returns are "reasonably normal" and the GRS test is robust to the existing non-normalities. Nevertheless, for other asset return series (for example, derivatives or high-frequency data), deviations from normality can be more severe. In these cases, we can use, for example, the asymptotic mean-variance efficiency test of MacKinlay and Richardson (1991) or Zhou's

(1993) generalization of the GRS test under an elliptical distribution. However, if returns do not follow an elliptical distribution, the economic meaning of the mean-variance criterion is not well-defined to begin with (see, e.g., Chamberlain, 1983) and in that case we would advise the use of more general stochastic dominance efficiency tests (see, e.g., Post, 2003; Kuosmanen, 2004).

Like the original GRS test, our test does not use conditioning information. There exists mounting evidence in favor of time-varying risk and time-varying risk aversion (see, e.g., Ferson *et al.*, 1987; Jagannathan and Wang, 1996; Lettau and Ludvigson, 2001). Conditional efficiency generally does not imply unconditional efficiency (see, e.g., Hansen and Richard, 1987), and conditional tests are needed in case of time variation. We refer to the recent paper of Ferson and Siegel (2006) for tests that use conditional information efficiently and generalizations of earlier work. On the other hand, given the lack of theoretical guidance for selecting the appropriate specification, conditional tests also entail risk of specification error (see, for example, Ghysels, 1998). In this paper we focus on unconditional efficiency and we leave the development of a conditional version of the test for future research. As a partial remedy, researchers and practitioners applying our unconditional test can use “ad hoc” approaches to control for time variation, including the formation of test portfolios that are periodically rebalanced, and moving or rolling window analysis.

Finally, we would also like to mention a number of other papers that are indirectly related to our work. The formulation of our test statistic for mean-variance efficiency is inspired by the work of Shapiro and Homem-de-Mello (1998). Higle and Sen (1991) and Shapiro and Homem-de-Mello (1998) derive general asymptotic tests for the optimality of a candidate solution to a stochastic optimization problem. De Roon, Nyman and Werker (2001) develop asymptotic tests for mean-variance spanning under short sale constraints and transaction costs, using a similar test statistic. We refer to Korkie and Turtle (2002) for an extensive mean-variance analysis of self-financing portfolios, including the derivation of spanning and efficiency tests under self-financing restrictions. Within our framework self-financing constraints can be imposed as well, but only on a sub-set of the risky assets.

The remainder of this study is structured as follows. Section 2 formulates the null hypothesis of mean-variance efficiency in an investment setting with a subset of restricted assets. Section 3 derives our generalization of the GRS test statistic and its small sample distribution. Section 4 analyzes the size and power of the test. Section 5 applies our test to two relevant practical cases: assessing portfolio efficiency in the presence of non-traded labor income and non-traded liabilities. Section 6 tests whether a value-weighted US stock-bond portfolio is mean-variance efficient, while taking into account a substantial position in non-traded human capital. Finally, Section 7 presents our conclusions and suggestions for further research. Throughout the text, we will use $\Re^N$ for an N-dimensional Euclidean space and $\Re_+^N$ for the positive orthant. To distinguish between vectors and scalars, we use a bold font for vectors and a regular font for scalars. Further, all vectors are column vectors and we use $\boldsymbol{r}'$ for the transpose of $\boldsymbol{r}$. Finally, $\mathbf{0}_N$ and $\mathbf{1}_N$ denote a (1xN) zero vector and a (1xN) unity vector.

2. Null Hypothesis of the Test

The investment universe includes N risky assets and a riskless asset.[1] Investors can construct portfolios $\boldsymbol{\lambda} \in \Re^N$. We assume that the first $N-R$ risky assets can be traded freely by the investor, but that trading of the last R assets is restricted, e.g. due to lack of liquidity. We split the portfolio weight vector $\boldsymbol{\lambda}' = [\boldsymbol{\lambda}_1' \quad \boldsymbol{\lambda}_2']$ up into the $N-R$ weights of the unconstrained assets $\boldsymbol{\lambda}_1 \in \Re^{N-R}$ and the R restricted weights $\boldsymbol{\lambda}_2 \in \Re^R$. The portfolio weights $\boldsymbol{\lambda}_2$ of the restricted assets are subject to a set of K equality constraints: $\mathbf{A}\boldsymbol{\lambda}_2 = \mathbf{b}$, with $\mathbf{A} \in \Re^{K \times R}$, $\mathbf{b} \in \Re^K$ and $K \leq R$. The restricted assets could for example include the investor's human capital (labor income), the investor's house or the liabilities of a pension fund. In these three cases the portfolio weight is typically fixed at a particular value: the constraint matrix then reduces to an identity matrix, i.e. $\mathbf{A} = \mathbf{I}_K$, while $\mathbf{b}$ specifies the values of the fixed portfolio weights, i.e. $\boldsymbol{\lambda}_2 = \mathbf{b}$. Assuming that the market is incomplete and no perfect hedge is available to undo the fixed portfolio weights, we will refer to these assets as "non-traded". The set of feasible portfolios is defined as $\boldsymbol{\Lambda} \equiv \left\{ \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix} \in \Re^N : \boldsymbol{\lambda}_1 \in \Re^{N-R}, \boldsymbol{\lambda}_2 \in \Re^R, \mathbf{A}\boldsymbol{\lambda}_2 = \mathbf{b} \right\}$. The special case $\mathbf{A} = \varnothing$ and $\mathbf{b} = \varnothing$, represents a test without restrictions on the portfolio weights, i.e. the traditional GRS test, while $\mathbf{A} = \mathbf{I}_K$ represents the special case with non-traded assets with given portfolio weights $\boldsymbol{\lambda}_2 = \mathbf{b}$.

Let $\boldsymbol{r} \in \Re^N$ denote the excess returns of the risky assets. The returns follow a joint distribution with mean $\boldsymbol{\mu} \equiv E[\boldsymbol{r}]$ and covariance matrix $\boldsymbol{\Omega} \equiv E[(\boldsymbol{r} - \boldsymbol{\mu})(\boldsymbol{r} - \boldsymbol{\mu})']$. We make a distinction between the expected excess returns of the traded assets $\boldsymbol{\mu}_1 \in \Re^{N\text{-}R}$ and the restricted assets $\boldsymbol{\mu}_2 \in \Re^R$, with $\boldsymbol{\mu} = \begin{bmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{bmatrix}$. We partition the covariance matrix similarly: $\boldsymbol{\Omega} = \begin{bmatrix} \boldsymbol{\Omega}_{11} & \boldsymbol{\Omega}'_{21} \\ \boldsymbol{\Omega}_{21} & \boldsymbol{\Omega}_{22} \end{bmatrix}$, with $\boldsymbol{\Omega}_{11} \in \Re^{(N-R)\times(N-R)}$ the covariance matrix of the traded assets, $\boldsymbol{\Omega}_{22} \in \Re^{R\times R}$ the covariance matrix of the restricted assets and $\boldsymbol{\Omega}_{21} \in \Re^{R\times(N-R)}$ collecting the covariance terms between the traded and restricted assets.

Investors choose investment portfolios to maximize a mean-variance objective function $g(r) = E[r] - \frac{1}{2}\zeta Var[r]$, where $\zeta \geq 0$ is a risk aversion parameter. The portfolio choice problem is

$$\max_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \{g(\boldsymbol{\lambda})\} = \max_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \{E[\boldsymbol{r}'\boldsymbol{\lambda}] - \tfrac{1}{2}\zeta Var[\boldsymbol{r}'\boldsymbol{\lambda}]\} = \max_{\boldsymbol{\lambda}\in\boldsymbol{\Lambda}} \{\boldsymbol{\mu}'\boldsymbol{\lambda} - \tfrac{1}{2}\zeta\boldsymbol{\lambda}'\boldsymbol{\Omega}\boldsymbol{\lambda}\} \tag{1}$$

A given portfolio $\boldsymbol{\tau} \in \Lambda$ is efficient if and only if it is an optimal solution of (1) and satisfies the first-order Karush-Kuhn-Tucker (KKT) conditions of the constrained optimization problem. Before we show the KKT conditions, we first define the alphas of the assets as the first-order derivatives of the objective function (1) with respect to the portfolio weights, evaluated at the given portfolio $\boldsymbol{\tau} \in \Lambda$:

$$\boldsymbol{\alpha} \equiv \left\{ \frac{d}{d\boldsymbol{\lambda}} g(\boldsymbol{\lambda}) \right\}_{\boldsymbol{\lambda}=\boldsymbol{\tau}} = \boldsymbol{\mu} - \zeta \boldsymbol{\Omega} \boldsymbol{\tau} \tag{2}$$

Using the expression for the alphas, the KKT optimality conditions for the efficiency of the given portfolio $\boldsymbol{\tau}$ are:

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_1 - \zeta(\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 + \boldsymbol{\Omega}'_{21}\boldsymbol{\tau}_2) \\ \boldsymbol{\mu}_2 - \zeta(\boldsymbol{\Omega}_{22}\boldsymbol{\tau}_2 + \boldsymbol{\Omega}_{21}\boldsymbol{\tau}_1) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N-R} \\ \mathbf{A}'\boldsymbol{\rho}_K \end{bmatrix} \tag{3}$$

with $\boldsymbol{\rho}_K \in \Re^K$ a vector of Lagrange multipliers for the K equality constraints on $\boldsymbol{\lambda}_2$. The KKT conditions are necessary and sufficient for the quadratic maximization problem subject to linear constraints (1), as the covariance matrix Ω is positive definite.[2]

In the unrestricted case the KKT conditions reduce to the familiar Euler equation $\boldsymbol{\alpha} = \mathbf{0}_N$, i.e. all alphas should equal zero. Note that in the case with restrictions, even if some restricted assets have non-zero alphas, the evaluated portfolio can still be mean-variance efficient. More specifically, the following polyhedral cone gives the set of admissible alphas:

$$C(\mathbf{A}) \equiv \{\mathbf{z} \in \Re^N : \mathbf{z} = \begin{bmatrix} \mathbf{0}_{N-R} \\ \mathbf{A}'\boldsymbol{\rho}_K \end{bmatrix}, \boldsymbol{\rho}_K \in \Re^K\} \tag{4}$$

This study develops a test for the null hypothesis that the evaluated portfolio is efficient, $H_0 : \boldsymbol{\alpha} \in C(\mathbf{A})$, against the alternative hypothesis of inefficiency, $H_1 : \boldsymbol{\alpha} \notin C(\mathbf{A})$. In the unrestricted case we find $C(\varnothing) = \mathbf{0}_N$ and the null reduces to $H_0 : \boldsymbol{\alpha} = \mathbf{0}_N$.

A remaining problem is the specification of the risk aversion parameter ζ of the investor holding portfolio $\boldsymbol{\tau}$. The GRS test implicitly chooses a value for this parameter by setting the alpha of the evaluated portfolio equal to zero, that is, $\boldsymbol{\alpha}'\boldsymbol{\tau} = 0$, which gives $\zeta_{GRS} \equiv (\boldsymbol{\mu}'\boldsymbol{\tau})(\boldsymbol{\tau}'\boldsymbol{\Omega}\boldsymbol{\tau})^{-1}$.[3] The alphas can then be expressed as $\boldsymbol{\alpha}_{GRS} \equiv \boldsymbol{\mu} - \zeta_{GRS}\boldsymbol{\Omega}\boldsymbol{\tau} = \boldsymbol{\mu} - (\boldsymbol{\mu}'\boldsymbol{\tau})\boldsymbol{\beta}$, with $\boldsymbol{\beta} \equiv (\boldsymbol{\Omega}\boldsymbol{\tau})(\boldsymbol{\tau}'\boldsymbol{\Omega}\boldsymbol{\tau})^{-1}$. This approach is generally not consistent with the null hypothesis in the case with restrictions on the portfolio weights, as the alpha of the evaluated portfolio does not necessarily has to equal zero. However, note that in the restricted case the alphas of the $N-R$ unrestricted assets still need to be zero: $\boldsymbol{\alpha}_1 = \mathbf{0}_{N-R}$. Hence, we can infer the investor's risk aversion parameter ζ from his portfolio of unrestricted assets, by solving the equation $\boldsymbol{\alpha}_1'\boldsymbol{\tau}_1 = 0$. This approach is consistent with the null hypothesis under restrictions and gives the following risk aversion parameter: $\zeta = (\boldsymbol{\mu}_1'\boldsymbol{\tau}_1)(\boldsymbol{\tau}_1'\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 + \boldsymbol{\tau}_1'\boldsymbol{\Omega}_{21}'\boldsymbol{\tau}_2)^{-1}$.

After substituting the expression for ζ in the KKT conditions (3), we obtain:

$$\boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\mu}_1 - (\boldsymbol{\mu}_1'\boldsymbol{\tau}_1)(\boldsymbol{\beta}_1'\boldsymbol{\tau}_1)^{-1}\boldsymbol{\beta}_1 \\ \boldsymbol{\mu}_2 - (\boldsymbol{\mu}_1'\boldsymbol{\tau}_1)(\boldsymbol{\beta}_1'\boldsymbol{\tau}_1)^{-1}\boldsymbol{\beta}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N-R} \\ \mathbf{A}'\boldsymbol{\rho}_K \end{bmatrix} \tag{5}$$

with the vector of betas defined as usual

$$\boldsymbol{\beta} = \boldsymbol{\Omega\tau}(\boldsymbol{\tau}'\boldsymbol{\Omega\tau})^{-1}, \quad \begin{bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{bmatrix} = \boldsymbol{\Omega\tau}(\boldsymbol{\tau}'\boldsymbol{\Omega\tau})^{-1} = \begin{bmatrix} (\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 + \boldsymbol{\Omega}'_{21}\boldsymbol{\tau}_2)(\boldsymbol{\tau}'\boldsymbol{\Omega\tau})^{-1} \\ (\boldsymbol{\Omega}_{22}\boldsymbol{\tau}_2 + \boldsymbol{\Omega}_{21}\boldsymbol{\tau}_1)(\boldsymbol{\tau}'\boldsymbol{\Omega\tau})^{-1} \end{bmatrix} \tag{6}$$

We will refer to the alphas defined by (5) as 'generalized alphas', because under portfolio weight restrictions they may differ from the classical alphas $\boldsymbol{\alpha}_{GRS}$. The relation between the generalized alphas and the GRS alphas is as follows: $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{GRS} + (\boldsymbol{\mu}'\boldsymbol{\tau} - \psi)\boldsymbol{\beta}$, with $\psi \equiv (\boldsymbol{\mu}_1'\boldsymbol{\tau}_1)(\boldsymbol{\beta}_1'\boldsymbol{\tau}_1)^{-1}$ the Treynor ratio of the portfolio of unrestricted assets.

3. Empirical Testing

An empirical test of mean-variance efficiency is based on a timeseries of risky asset excess returns $\boldsymbol{r}_t$ observed at time $t = 1, \cdots, T$, where $\boldsymbol{r}_t \in \Re^N$ is a (Nx1) vector of returns. By analogy to GRS, we define the data generating process (DGP) as

$$\boldsymbol{r}_t = \boldsymbol{\alpha}_{GRS} + \boldsymbol{\beta}(\boldsymbol{r}_t'\boldsymbol{\tau}) + \boldsymbol{\varepsilon}_t, \; t = 1, \cdots, T \tag{7}$$

We assume that the regression errors $\boldsymbol{\varepsilon}_t$ are serially independent and identically distributed random draws from a multivariate normal distribution with mean $\mathbf{0}_N$ and covariance matrix $\boldsymbol{\Sigma}_\varepsilon \in \Re^{N\times N}$, conditional on the returns $(\boldsymbol{r}_t'\boldsymbol{\tau})$ of the investor's portfolio at time t. Least squares estimation of the DGP (7) gives estimates of the classical betas $\boldsymbol{\beta}$ and alphas $\boldsymbol{\alpha}_{GRS}$, but not an estimate of the generalized alphas $\boldsymbol{\alpha}$ under restrictions. To estimate the generalized alphas we use the relation $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{GRS} + (\boldsymbol{\mu}'\boldsymbol{\tau} - \psi)\boldsymbol{\beta}$, and replace $\boldsymbol{\alpha}_{GRS}$ in (7) by $\boldsymbol{\alpha} - (\boldsymbol{\mu}'\boldsymbol{\tau} - \psi)\boldsymbol{\beta}$. After some rearranging of the terms, we find:

$$\boldsymbol{r}_t = \boldsymbol{\alpha} + \boldsymbol{\beta}(\boldsymbol{\beta}_1'\boldsymbol{\tau}_1)^{-1}(\boldsymbol{r}_{1,t}'\boldsymbol{\tau}_1) + \boldsymbol{u}_t,\ t = 1,\cdots,T \tag{8}$$

with the error term $\boldsymbol{u}_t$ defined as

$$\boldsymbol{u}_t = \boldsymbol{\varepsilon}_t + \boldsymbol{\beta}(\boldsymbol{r}_t'\boldsymbol{\tau} - \boldsymbol{\mu}'\boldsymbol{\tau}) + \boldsymbol{\beta}(\boldsymbol{\beta}_1'\boldsymbol{\tau}_1)^{-1}(\boldsymbol{\mu}_1'\boldsymbol{\tau}_1 - \boldsymbol{r}_{1,t}'\boldsymbol{\tau}_1) \quad , t = 1,\ldots,T. \tag{9}$$

The error term $\boldsymbol{u}_t$ follows a multivariate normal distribution with $E[\boldsymbol{u}_t] = 0$, conditional on the returns $(\boldsymbol{r}_t'\boldsymbol{\tau})$ of the investor's entire portfolio $\boldsymbol{\tau}$ – including the R restricted assets – and the returns $(\boldsymbol{r}_t'\boldsymbol{\kappa})$ on the portfolio $\boldsymbol{\kappa} = \begin{bmatrix} \boldsymbol{\tau}_1 \\ \mathbf{0}_R \end{bmatrix}$ of $N - R$ unrestricted assets. We define the covariance matrix of the regression errors as $\boldsymbol{\Sigma} = E[\boldsymbol{u}_t\boldsymbol{u}_t']$.

Given the estimated betas $\hat{\boldsymbol{\beta}}$, we propose the following unbiased estimator for the generalized alphas based on (8):

$$\hat{\boldsymbol{\alpha}} \equiv \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}_1'\boldsymbol{\tau}_1)^{-1}(\hat{\boldsymbol{\mu}}_1'\boldsymbol{\tau}_1). \tag{10}$$

Since the errors are jointly normally distributed, the estimated generalized alphas also follow a joint normal distribution, conditional on the returns of the portfolios $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$:

$$\hat{\boldsymbol{\alpha}} \sim N(\boldsymbol{\alpha}, T^{-1}(1 + \hat{\theta}^2)\boldsymbol{\Sigma}) \tag{11}$$

with $\hat{\theta} = \hat{S}_\kappa \hat{\rho}_{\boldsymbol{\tau\kappa}}^{-1}$, where $\hat{S}_\kappa$ is the Sharpe ratio of the unrestricted asset portfolio $\boldsymbol{\kappa}$, and $\hat{\rho}_{\boldsymbol{\tau\kappa}}$ the estimated correlation between the returns of the portfolio $\boldsymbol{\tau}$ – including the R restricted assets – and the returns of the unrestricted asset portfolio $\boldsymbol{\kappa}$. The full derivation of (11) is in the Appendix.[4]

As a test statistic, we will use the smallest distance between the estimated generalized alphas and the cone of admissible alphas (4):

$$\xi(\mathbf{A}) \equiv \min_{\boldsymbol{z} \in C(\mathbf{A})} (1+\hat{\theta}^2)^{-1} (\hat{\boldsymbol{\alpha}} - \boldsymbol{z})' \hat{\boldsymbol{\Sigma}}^{-1} (\hat{\boldsymbol{\alpha}} - \boldsymbol{z}) \tag{12}$$

where $\hat{\boldsymbol{\Sigma}} \equiv (T-2)^{-1} (\sum_{t=1}^{T} \hat{\boldsymbol{u}}_t \hat{\boldsymbol{u}}_t')$ is an unbiased estimator of $\boldsymbol{\Sigma}$, based on the empirical regression errors $\hat{\boldsymbol{u}}_t \equiv \boldsymbol{r}_t - \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\beta}} (\hat{\boldsymbol{\beta}}' \boldsymbol{\kappa})^{-1} (\boldsymbol{r}_t' \boldsymbol{\kappa})$. The test statistic is a restricted version of the classical Hotelling's T^2 statistic used in multivariate statistical analysis.

GRS derive the small sample distribution of the unrestricted test statistic $\xi(\varnothing) = (1+\hat{\theta}^2)^{-1} \hat{\boldsymbol{\alpha}}_{GRS}' \hat{\boldsymbol{\Sigma}}^{-1} \hat{\boldsymbol{\alpha}}_{GRS}$. The estimates $\hat{\boldsymbol{\alpha}}_{GRS}$ and $(T-2)\hat{\boldsymbol{\Sigma}}$ are independent and follow the normal distribution in (11) and a Wishart distribution with parameter matrix $\boldsymbol{\Sigma}$ and $(T-2)$ degrees of freedom, respectively. It follows that a simple transformation of the test statistic follows an F-distribution:

$$\left(\frac{T(T-N-1)}{N(T-2)} \right) \xi(\varnothing) \sim f_{N,(T-N-1),\lambda_{\varnothing}} \tag{13}$$

with non-centrality parameter

$$\lambda_{\varnothing} \equiv \tfrac{1}{2} T (1+\hat{\theta}^2)^{-1} \boldsymbol{\alpha}_{GRS}' \boldsymbol{\Sigma}_{\varepsilon}^{-1} \boldsymbol{\alpha}_{GRS} \tag{14}$$

For the case with R restricted assets, we will now derive the exact small sample distribution of the test statistic. We define the augmented constraint matrix $\overline{\mathbf{A}} \in \Re^{K\times N}$ as $\overline{\mathbf{A}} = \lfloor \mathbf{O}_{K,N-R} \quad \mathbf{A} \rfloor$, where $\mathbf{O}_{K,N-R} \in \Re^{K\times(N-R)}$ denotes a zero matrix. The null hypothesis is $H_0 : \boldsymbol{\alpha} = \begin{bmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{N-R} \\ \mathbf{A}'\boldsymbol{\rho}_K \end{bmatrix} = \overline{\mathbf{A}}'\boldsymbol{\rho}_K$. Let $\mathbf{M} \in \Re^{N\times(N-K)}$ denote a matrix whose columns form a basis set for the null space of $\overline{\mathbf{A}}$. Note that the range of the matrix $\mathbf{M}$, denoted by $R(\mathbf{M})$, is equal to the null space of the matrix $\overline{\mathbf{A}}$, denoted by $N(\overline{\mathbf{A}})$: $R(\mathbf{M}) = N(\overline{\mathbf{A}})$. According to the fundamental theorem of linear algebra, $N(\mathbf{M}') = R(\overline{\mathbf{A}}')$. The null hypothesis, $H_0 : \boldsymbol{\alpha} \in R(\overline{\mathbf{A}}')$, is therefore equivalent to $H_0 : \boldsymbol{\alpha} \in N(\mathbf{M}')$, i.e. $H_0 : \mathbf{M}'\boldsymbol{\alpha} = \mathbf{0}_{N-K}$. We can now formulate the test statistic as follows (see the Appendix for the full derivation):

$$\xi(\mathbf{A}) = (1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}'\mathbf{M}(\mathbf{M}'\hat{\boldsymbol{\Sigma}}\mathbf{M})^{-1}\mathbf{M}'\hat{\boldsymbol{\alpha}} \tag{15}$$

The vector $\mathbf{M}'\hat{\boldsymbol{\alpha}}$ follows a (N – K)-dimensional multivariate normal distribution with mean $\mathbf{M}'\boldsymbol{\alpha}$ and covariance matrix $T^{-1}(1+\hat{\theta}^2)\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M}$. Hence, the distribution of $\xi(\mathbf{A})$ is known:

$$\left(\frac{T(T-N+K-1)}{(N-K)(T-2)}\right)\xi(\mathbf{A}) \sim f_{(N-K),(T-N+K-1),\lambda_{\mathbf{A}}} \tag{16}$$

with non-centrality parameter

$$\lambda_{\mathbf{A}} \equiv \tfrac{1}{2}T(1+\hat{\theta}^2)^{-1}\boldsymbol{\alpha}'\mathbf{M}(\mathbf{M}'\boldsymbol{\Sigma}\mathbf{M})^{-1}\mathbf{M}'\boldsymbol{\alpha} \tag{17}$$

Under the null hypothesis, $\lambda_{\mathbf{A}} = 0$ and the test statistic follows a central F-distribution with $(N - K)$ and $(T - N + K - 1)$ degrees of freedom.

The most relevant applications of our efficiency test under restrictions involve non-traded assets with a fixed portfolio weight, such as the investor's labor income, housing or the liabilities of a pension fund. For these applications the portfolio weight restrictions are $\boldsymbol{\lambda}_2 = \mathbf{b}$, $K=R$, and the constraint matrix reduces to an identity matrix: $\mathbf{A} = \mathbf{I}_R$. Given the simple structure of the constraint matrix, it is straightforward to show that $\mathbf{M} = \begin{bmatrix} \mathbf{I}_{N-R} \\ \mathbf{O}_{R,N-R} \end{bmatrix}$. Note that $\mathbf{M}'\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\alpha}}_1$ and $\mathbf{M}'\hat{\mathbf{\Sigma}}\mathbf{M} = \hat{\mathbf{\Sigma}}_{11}$, and therefore the test statistic $\xi(\mathbf{I}_R)$ reduces to

$$\xi(\mathbf{I}_R) = (1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}_1'\hat{\mathbf{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\alpha}}_1 \tag{18}$$

Hence, for the special case of non-traded assets, the expression for the test statistic can be simplified considerably. At first sight, it might appear that the alphas $\hat{\boldsymbol{\alpha}}_1$ and regression errors for the $N - R$ unrestricted assets determine the value of the test statistic completely, while the R non-traded assets play no obvious role. Note, however, that the estimated alphas $\hat{\boldsymbol{\alpha}}_1$ of the unrestricted assets depend explicitly on the covariance between the excess returns of the unrestricted assets and the non-traded assets. The same holds for $\hat{\theta}$ and $\hat{\mathbf{\Sigma}}_{11}$.

4. Size and Power of the Test

We will now investigate the size and power of our efficiency test under restrictions. The small sample distribution of the test statistic in this paper – and in GRS – is derived under the assumption of a conditional multivariate normal return distribution, given the returns of the portfolio that we would like to assess. An *unconditional* multivariate normal distribution for the asset returns, treating the returns of the given portfolio as a function of the random individual asset returns, i.e. as a random variable, seems more appropriate. Fortunately, Jobson and Korkie (1985) show that the GRS test-statistic follows an F-distribution as well under the assumption of an unconditional multivariate normal return distribution. Further, numerical results in Jobson and Korkie (1982) and Campbell et al. (1997) demonstrate that the GRS test performs much better in a multivariate normal setting – in terms of size and power – than alternative asymptotic tests of portfolio efficiency, such as the Wald test statistic of Jobson and Korkie (1982, JK) . Given that our efficiency test is an extension of GRS, a priori we would expect our test to perform well in small samples, regardless of whether the underlying return distribution is conditionally or unconditionally normal. On the other hand, the asymptotic test for efficiency under restrictions proposed by Gouriéroux and Jouneau (1999) is an extension of the asymptotic Wald test of JK, and for this reason we do not expect it to perform well in small samples. We will now conduct simulation experiments to verify these premises.

As a starting point for the simulation we use stock and bond return data from the US, consisting of the Ibbotson long-term government bond index, the Ibbotson long-term corporate bond index and six Fama and French portfolios resulting from a two by three double-sorting of all US stocks based on size and value (source: homepage of Kenneth French). We refer to Table 1 for descriptive statistics of annual total return data from the period 1956-2005. After estimating the sample mean and covariance matrix of the returns, we calculate the weights of the unconstrained ex post tangency portfolio (w_u) with maximum Sharpe ratio. As an example of weight constraints, we fix the portfolio weight of long-term government bonds at 40% and the weight of long-term corporate bonds at 20%. We calculate the weights of the ex post tangency portfolio subject to these constraints (w_c). Next, we draw random samples of length T from a unconditional multivariate normal distribution with mean and covariance matrix fixed at the sample values. We calculate the returns of the unconstrained portfolio w_u and test its efficiency with the GRS F statistic and the JK Wald χ^2 statistic. We calculate the returns of the constrained portfolio w_c and test its efficiency with the F test derived in this paper and the χ^2 test of Gouriéroux and Jouneau (1999). The simulation is repeated a total of S times to replicate the empirical distribution of the test statistics.

One important difference in the implementation of the F tests and the χ^2 tests is that the χ^2 tests ideally should include all primary assets that are part of the given portfolio, as otherwise the test statistic erroneously could take on negative values. On the other hand, the F tests should never use all primary assets in the given portfolio as test assets, as in that case the residual covariance matrix $\sum$ of the regressions in (7) and (8) is singular and the test statistic cannot be computed. For this reason we use all N = 8 primary assets (two bond portfolios and six FF portfolios) to implement the χ^2 tests, while we calculate the F tests with N = 6 primary assets, excluding the mid-cap value portfolio and the mid-cap size portfolio from the set of FF portfolios. Overall, the unconditional normal simulation setting with known optimal portfolio weights favors the Wald χ^2 tests, as both the JK and GJ test were derived under these assumptions.[5]

We also assess the power of the various tests in the simulation runs. For this purpose we test the efficiency of an equally weighted portfolio of the unrestricted assets, which is clearly inefficient based on the ex post Sharpe ratio. Table 3 shows the results of the simulations. For each test the columns of the table show the mean and the variance of the simulated test statistic, the size of the test at the 1%, 5% and 10% level (rejection rate of the ex post efficient portfolio) and the power of the test at the 1%, 5% and 10% level (rejection rate of the equally weighted portfolio). Directly below each row of simulation results we show the mean and variance of the theoretical test statistic distribution for comparison. With a small sample size of T = 50 observations, the size of the GRS F test is nearly identical to the pre-set significance level, while the JK Wald test has a much larger Type I error (13.8% at the 5% significance level and 5.2% at the 1% level). For the tests under portfolio weight constraints we find similar results: with T = 50 observations the F test derived in this paper has a size that is very close to the desired significance level, while the Wald test of Gouriéroux and Jouneau (1999) rejects the null hypothesis too often (e.g., a 10.7% rejection rate at the 5% significance level). At small sample sizes, i.e. T = 50 and T = 100, the F tests perform much better than in terms of size than the Wald tests, while in larger samples (T = 200, T = 400 and T = 800) the performance of the Wald test gradually improves.

The power of the Wald tests is generally slightly higher than the power of the F tests in small samples, but this is not a big advantage, given the corresponding large Type I error: the Wald tests reject the null hypothesis more often, regardless of whether the null is true or not. In samples of T = 200 and larger, the power of the F and Wald tests is similar. Please note that the estimated mean and variance in Table 3 indicate that the simulated distribution of our test statistic follows the theoretical F distribution closely in small samples. This is not the case for the Gouriéroux and Jouneau (1999) statistic, which has a much higher mean and variance in small samples than the theoretical (asymptotic) χ^2 distribution.

Overall, these simulation results indicate that our F test for mean-variance efficiency with non-traded assets has similar favorable properties as the GRS test in small samples, performing better than the asymptotic Wald test of Gouriéroux and Jouneau (1999). Further, if run the Wald tests with a reduced set of N = 6 primary assets, instead of the complete set of 8 assets, then the χ^2 test statistic is not well-defined (can become negative) and the simulated distribution becomes completely different from theoretical distribution, with poor simulated test size results (results not reported to save space, but available upon request).

5. Testing the Efficiency of Portfolios with Non-Traded Assets

In this section we show how our test for portfolio efficiency with restricted assets can be applied in the presence of non-traded liabilities, as well as in the case of non-traded labor income. We do not discuss the relevant case of a non-traded position in housing to save some space, but the approach follows the same steps as in the two examples in this section.

5.1 Asset-Liability Management

An interesting application of our mean-variance test under restrictions is to test the efficiency of portfolios that are evaluated relative to an exogenous, non-traded, stochastic benchmark. For example, the risk and return of the investment portfolio of a defined benefit pension plan are usually measured relative to the growth of the plan liabilities L_t, defined as the net present value of all future pension payments. The plan surplus, S_t, is defined as the difference between the value of the assets, A_t, and the liabilities: $S_t = A_t - L_t$. Given a fixed level of plan contributions, in the short-term the fund managers of the plan typically make a trade-off between maximizing the expected value of the plan surplus $E[S_{t+1}]$ and avoiding unpredictable fluctuations in the surplus that might lead to plan deficits ($S_{t+1} < 0$). This trade-off can be formalized with the following mean-variance surplus management problem:

$$\max E[S_{t+1}] - \tfrac{1}{2}\zeta Var[S_{t+1}] \tag{19}$$

Let's assume for ease of exposition that a risk free asset with return R_0 exists. Let r_L denote the random return on the liabilities from time t to t+1, in excess of the risk free rate. The (Ix1) vector $\boldsymbol{r}_1$ denote the excess returns of a set of unrestricted risky assets available to the pension fund portfolio manager. Given the (Ix1) vector of investment portfolio weights $\boldsymbol{\lambda}_1$, the surplus at time t+1 is equal to $S_{t+1} = (1+\boldsymbol{r}_1'\boldsymbol{\lambda}_1 + R_0)A_t - (1+r_L + R_0)L_t$

$= (1+R_0)S_t + A_t(\mathbf{r}_1'\boldsymbol{\lambda}_1 - (L_t / A_t)r_L)$. The mean-variance surplus management problem can now be reduced to the following equivalent formulation:

$$\max E[\mathbf{r}_1'\boldsymbol{\lambda}_1 - (L_t / A_t)r_L] - \tfrac{1}{2}\tilde{\zeta}Var[\mathbf{r}_1'\boldsymbol{\lambda}_1 - (L_t / A_t)r_L] \tag{20}$$

with $\tilde{\zeta} = A_t\zeta$. The surplus management problem as defined above in (20) has been proposed and studied by Sharpe and Tint (1990).

Suppose that the plan manager would like to evaluate the mean-variance "surplus efficiency" of the given (*I*x1) risky asset portfolio $\boldsymbol{\tau}_1$, assuming no constraints on the risky asset weights. The first order conditions for mean-variance surplus efficiency of $\boldsymbol{\tau}_1$ are:

$$\boldsymbol{\alpha}_1 \equiv \boldsymbol{\mu}_1 - \tilde{\zeta}\left(\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 - (L_t / A_t)\boldsymbol{\sigma}_L\right) = \mathbf{0}_I \tag{21}$$

where the (*I*x1) row vector $\boldsymbol{\mu}_1$ denotes expected excess returns of the risky assets and $\boldsymbol{\Omega}_{11}$ the corresponding (*I*x*I*) covariance matrix, while the (*I*x1) vector $\boldsymbol{\sigma}_L$ measures the return covariance between the risky assets and the liabilities.

So far, the risk aversion parameter $\tilde{\zeta}$ has not been specified yet. To give the plan's fund manager the benefit of the doubt, we set the value of $\tilde{\zeta}$ such that the evaluated portfolio $\boldsymbol{\tau}_1$ has zero alpha. i.e. $\tilde{\zeta} = \boldsymbol{\mu}_1'\boldsymbol{\tau}_1 / [\boldsymbol{\tau}_1'\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 - (L_t / A_t)\boldsymbol{\sigma}_L'\boldsymbol{\tau}_1]$. The first-order efficiency condition now is

$$\boldsymbol{\alpha}_1 \equiv \boldsymbol{\mu}_1 - \boldsymbol{\mu}_1'\boldsymbol{\tau}_1 \frac{(\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 - (L_t / A_t)\boldsymbol{\sigma}_L)}{(\boldsymbol{\tau}_1'\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 - (L_t / A_t)\boldsymbol{\sigma}_L'\boldsymbol{\tau}_1)} = \mathbf{0}_I \tag{22}$$

Please note the equivalence between the first order conditions for the $N-R$ unrestricted assets in (5) and the first order conditions of the surplus management problem (22) above. Our mean-variance test under restrictions can be applied to derive an unbiased estimator for the alphas $\boldsymbol{\alpha}_1$ and a multivariate test statistic. Within our framework we simply treat the I risky assets as $N-R$ unrestricted assets with weights $\boldsymbol{\lambda}_1$ and excess returns $\boldsymbol{r}_1$, and the plan liabilities as a single restricted asset, i.e. with $R = 1$, with excess return $\boldsymbol{r}_2 = r_L$ and portfolio weight $\boldsymbol{\lambda}_2$. The plan's short position in the liabilities can be modeled with the single equality constraint $\mathbf{A}\boldsymbol{\lambda}_2 = \mathbf{b}_2$ with $\mathbf{A} = 1$ and $\mathbf{b} = -(L_t / A_t)$. Note that $N = I + 1$ and $K = 1$.

To implement the empirical test, we first estimate the traditional market model (7) relative to the returns $(\boldsymbol{r}_t'\boldsymbol{\tau})$ on the pension fund's augmented portfolio $\boldsymbol{\tau}' = \left[\boldsymbol{\tau}_1' \quad -(L_t / A_t)\right]'$:

$$\boldsymbol{r}_{1,t} = \boldsymbol{\alpha}_{1,GRS} + \boldsymbol{\beta}_1(\boldsymbol{r}_t'\boldsymbol{\tau} - (L_t / A_t) r_{L,t}) + \boldsymbol{\varepsilon}_{1,t},\ t = 1,\cdots,T \tag{23}$$

Next, the generalized alphas are estimated with equation (10):

$$\hat{\boldsymbol{\alpha}}_1 = \hat{\boldsymbol{\mu}}_1 - \hat{\boldsymbol{\beta}}_1(\hat{\boldsymbol{\beta}}_1'\boldsymbol{\tau}_1)^{-1}(\hat{\boldsymbol{\mu}}_1'\boldsymbol{\tau}_1) \tag{24}$$

We calculate the residuals $\hat{\boldsymbol{u}}_{1,t}$ corresponding to (24) and estimate the covariance matrix $\hat{\boldsymbol{\Sigma}}_{11}$. Next, we compute the value of test statistic as $\xi = (1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\alpha}}_1$, with $\hat{\theta} = \hat{S}_1\hat{\rho}_{1\boldsymbol{\tau}}^{-1}$, where $\hat{S}_1$ is the Sharpe ratio of the risky asset portfolio $\boldsymbol{\tau}_1$ and $\hat{\rho}_{1\boldsymbol{\tau}}$ the estimated correlation between the returns of the augmented portfolio $\boldsymbol{\tau}$ – including the short position in the liabilities – and the returns of the risky asset portfolio $\boldsymbol{\tau}_1$. The test statistic for the mean variance surplus efficiency of portfolio $\boldsymbol{\tau}_1$ follows an F-distribution with $(N-K)=I$ and $(T-N+K-1)$ $=(T-I-1)$ degrees of freedom.

5.2 Mean-Variance Test with Non-Traded Labor Income

A second relevant application of our mean-variance test under restrictions is to test the efficiency of portfolios of individuals with non-tradable labor income. We consider a non-retired individual investor. At time t the individual's overall wealth W_t consists of a liquid investment portfolio A_t – invested in bonds, stocks, etc .. – and the expected net present value of future labor income, denoted by Y_t.[6] The net present value of labor income at time t+1 is defined as: $Y_{t+1} = (1 + r_Y + R_0)Y_t$, with r_Y a normally distributed random variable. The (Ix1) vector $\boldsymbol{r}_1$ denotes the excess returns on the risky assets available to the individual, following a multivariate normal distribution. Given the (Ix1) vector of investment portfolio weights $\boldsymbol{\lambda}_1$, the individual's wealth at time t+1 is $W_{t+1} = A_{t+1} + Y_{t+1} = (1+\boldsymbol{r}_1'\boldsymbol{\lambda}_1 + R_0)A_t + (1+r_Y+R_0)Y_t$ $=(1+R_0)W_t + ((A_t/W_t)\boldsymbol{r}_1'\boldsymbol{\lambda}_1 + (Y_t/W_t)r_Y)W_t$.

The individual investor's aim is to invest in an efficient portfolio in terms of wealth at time t+1,

$$\max E[W_{t+1}] - \tfrac{1}{2}\zeta Var[W_{t+1}] \tag{25}$$

which is equivalent to the maximizing the following objective,

$$\max E[(A_t/W_t)\boldsymbol{r}_1'\boldsymbol{\lambda}_1 + (Y_t/W_t)r_Y] - \tfrac{1}{2}\tilde{\zeta} Var[(A_t/W_t)\boldsymbol{r}_1'\boldsymbol{\lambda}_1 + (Y_t/W_t)r_Y] \tag{26}$$

with $\tilde{\zeta} = W_t\zeta$.

Following the same steps as before, we can derive the first order conditions for the mean-variance efficiency of a given (Ix1) risky asset portfolio $\boldsymbol{\tau}_1$:

$$\boldsymbol{\alpha}_1 \equiv \boldsymbol{\mu}_1 - \boldsymbol{\mu}_1'\boldsymbol{\tau}_1 \frac{((A_t/W_t)\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 + (Y_t/W_t)\boldsymbol{\sigma}_Y)}{((A_t/W_t)^2\boldsymbol{\tau}_1'\boldsymbol{\Omega}_{11}\boldsymbol{\tau}_1 + (A_t/W_t)(Y_t/W_t)\boldsymbol{\sigma}_Y'\boldsymbol{\tau}_1)} = \mathbf{0}_I \tag{27}$$

with the (Ix1) vector $\boldsymbol{\sigma}_Y$ measuring the covariance between the excess asset return and the change in the present value of labor income.

Our methodology can be applied to derive an unbiased estimator for the alphas $\boldsymbol{\alpha}_1$ and a multivariate test statistic for the mean-variance efficiency of the portfolio $\boldsymbol{\tau}_1$, given the investor's non-tradable labor income. Within our framework we treat the I risky assets as N–R unrestricted assets with portfolio weights $\boldsymbol{\lambda}_1$ and we take the net value present value of labor income as a single restricted asset $\boldsymbol{\lambda}_2$, subject to the constraint $\boldsymbol{\lambda}_2 = Y_t/W_t$. To estimate the classical betas $\hat{\boldsymbol{\beta}}_1$ we use the market model (7) relative to the returns on the individual's overall portfolio – including the value of labor income – and we use equation (10) to estimate the generalized alphas $\hat{\boldsymbol{\alpha}}_1$. The test statistic, $\xi = (1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}_1'\hat{\boldsymbol{\Sigma}}_{11}^{-1}\hat{\boldsymbol{\alpha}}_1$, follows an F-distribution with I and $(T-I-1)$ degrees of freedom.

6. Empirical Application

In this section we will illustrate our mean-variance efficiency test under restrictions with an empirical application. We will examine US stock market data to test if a proxy for the market portfolio is mean-variance efficient for an individual investor with labour income. For various reasons, market portfolio efficiency is an interesting hypothesis. First, the Sharpe-Lintner-Mossin CAPM predicts that the market portfolio is efficient. Second, market portfolio efficiency seems consistent with the popularity of passive mutual funds and exchange traded funds that track broad value-weighted indexes.

As a proxy for the market portfolio we construct a portfolio that invests 50% in US bonds and 50% in the CRSP all-share index, which is the value-weighted average of all common stocks listed on the NYSE, AMEX and NASDAQ markets and covered by CRSP. The 50% portfolio weight of bonds consists of an investment of 25% in long-term US government bonds and 25% in long-term corporate bonds, both represented by total return indices of Ibbotson and Associates. The 50% percent portfolio weight that we assign to bonds is not based on prior information about the total market value of US long-term bonds relative to the total market value of US equity, but serves as an example and crude approximation.

We use two sets of test assets. The first set consists of 12 value-weighted industry portfolios from the data library on the homepage of Kenneth French. The second set of test assets consists of the Ibbotson long-term government bond index, the Ibbotson long-term corporate bond index and four Fama and French portfolios: small stocks with low price to book (SL), small stocks with high price to book (SH), big stocks with low price to book (BL) and big stocks with high price to book (BH). The four Fama and French portfolio were selected from six portfolios that result from a two by three double-sorting of all US stocks based on size and value, available from the data library on the homepage of Kenneth French.

We use annual return data from the post-war era 1956-2005, a total of 50 observations. We use annual data for three reasons. First, as argued by Benartzi and Thaler (1995), we expect that many investors have an investment horizon of one year. Second, we know the exact distribution of the test statistic under normality and we would like to exploit this advantage of the test in a small sample setting. Third, annual returns follow a normal distribution more closely than asset returns of higher frequency (e.g. monthly, weekly or daily returns), which is important given that we assumed normality to derive the distribution of the test statistic. Table 1 and Table 2 display descriptive statistics of the excess return series, as well as the estimated correlations between the excess returns.

6.1 Mean-Variance Test Results with Non-Traded Labor Income

We first test the efficiency of our proxy for the market portfolio, consisting of 50% bonds and 50% stocks, relative to the set of 12 industry portfolios with the unconstrained GRS test, without taking the individual's labor income into account. Table 4 shows that the p-value of the unconstrained GRS test is equal to 0.297, indicating that the efficiency of the given portfolio cannot be rejected.

We now additionally take into account the estimated value of the individual's labor income, assuming that it cannot be hedged perfectly and that its weight in total wealth is fixed at Y_t/W_t. For the growth rate of the individual's labor income we use the yearly change in the series "Average hourly earnings of production workers" in the manufacturing sector from the US Bureau of Labor Statistics (http://www.bls.gov/). Table 1 and 2 show descriptive statistics for this series. We choose this particular series mainly as an illustration, expecting it to capture systematic fluctuations in labor income in the sector that are relevant for portfolio choice. We would like mention for the sake of completeness that individual labor income has a volatile idiosyncratic component – due to the career path of the individual – that is not fully captured in an average hourly earnings series. We refer readers interested in a careful panel estimation of the individual labor income process to Cocco, Gomes and Maenhout (2005).

What value should we give to Y_t/W_t, the present value of labor income divided by total wealth? This ratio will vary strongly from one individual to another, but some quick back-of-the-envelope calculations show that the present value of labor income will dominate other sources of wealth for most wage-earners. For example, consider a relatively wealthy individual, 10 years from retirement, with a liquid investment portfolio of $800,000 (assume no homeownership for the sake of simplicity) and an annual income of \$100,000 growing at 3% per year on average. Setting the discount rate for future labor income at 5%, we find that the present value of labor income is \$901,002, and the ratio Y_t/W_t = 53%. Considering the same individual at 5 or 15 years from retirement, the ratios are Y_t/W_t = 37% and Y_t/W_t = 62%, respectively. For a young individual, 40 years from retirement, with an annual income of \$30,000 and an initial asset portfolio of \$150,000, the ratio Y_t/W_t is 89%. At 15 and 25 years from retirement, the ratio is 80% and 86%, respectively. Given these and similar estimates, we expect the labor-to-total-wealth ratio to be relatively high for the typical (median) wage-earner and we use 50%, 70% and 90% as base cases for Y_t/W_t.

Table 4 shows the test results for the efficiency of the stock-bond market portfolio proxy, relative to the 12 industry portfolios, given a restricted "investment" in labor income – growth rate based on the BLS manufacturing average earnings – with Y_t/W_t equal to 50%, 70% and 90%, respectively. In all three cases we find that efficiency of the given portfolio cannot be rejected (p-values 0.292, 0.293 and 0.612), as in the unrestricted GRS case without labor income. Interestingly, though, the estimated alphas of some of the 12 industry portfolios change considerable once labor income is taken into account. For example, as the present value of labor income from working in the manufacturing industry becomes a larger component of the individual's total wealth, the estimated alpha of the Manufacturing industry portfolio turns from positive (0.3% per year) to strongly negative (-3.4% per year). This effect arises due to the relatively high correlation of 0.19 between the excess returns of the Manufacturing industry portfolio and the change of average hourly earnings in the industry (measured in excess of the risk-free rate), reported in Table 1. We find a similar positive correlation, and hence decreasing alpha at higher levels of Y_t/W_t, for the industry portfolios Business Equipment, Durable Consumer Goods and Energy. For the industry portfolios Non-

Durable Consumer Goods, Health Care and Telecommunications, on the other hand, the correlation is negative and the alpha increases at higher levels of Y_t/W_t.

Table 4 shows that the average deviation of the estimated alphas from the null hypothesis value of zero – defined as $\sqrt{\sum \hat{\alpha}_i^2}$ – is relatively high at Y_t/W_t = 0.90, but on the other hand the value of test statistic is relatively low and the null hypothesis cannot be rejected (p-value 0.612). Basically, as the value of labor income starts to dominate the individual's total wealth, the standard deviation of the regression errors associated with the estimated alphas become larger, as wage growth is not very strongly correlated with the industry portfolio returns. This latter "increasing error" effect dominates the increase in the deviation of the alphas from zero and overall the value of the test statistic decreases, leading to lower test significance.

In Table 5 we repeat the efficiency tests, using as test assets the Ibbotson long-term government bond index, the Ibbotson long-term corporate bond index and four Fama and French value/size portfolios (Fama and French, 1992). Not surprisingly, due to the presence of strong size and value effects in this set of returns, the unconstrained GRS test – without considering labor income – strongly rejects the efficiency of our stock-bond market portfolio proxy (p-value of 0.001). The small value portfolio sticks out with an estimated alpha of 7.9% per annum, followed at some distance by the portfolio of large-cap value stocks with an alpha of 3.6% per annum. After taking into account the individual's labor income, the estimated alpha of the small value portfolio shrinks to 3.0% per year at a labor-income-to-total-wealth ratio of Y_t/W_t = 0.90. At the same time, the estimated alpha of the small growth company portfolio (SL) drops sharply from -0.4% in the GRS-case to -8.3% in the presence of labor income (at Y_t/W_t = 0.90). Further, bonds become more attractive in the presence of labor income. These effects are driven by the positive correlation between the returns of small stocks – both value and growth – and the growth of average hourly earnings in the manufacturing industry (measured in excess of the risk-free rate) and the negative correlation between bonds returns with labor income growth. Although the estimated alphas change in the presence of labor income, the overall deviation of the estimated alphas from zero remains of similar magnitude and efficiency of the market portfolio proxy is strongly rejected. In the case $Y_t/W_t = 0.90$ the regression errors of the alpha estimates are relatively high, leading to a somewhat higher p-value of 0.019 for the efficiency test, but efficiency is still clearly rejected at the 5% level.

7. Conclusions

This paper extends the classical Gibbons, Ross and Shanken (1989) test for mean-variance efficiency of a given portfolio to include linear equality restrictions on the weights of a subset of restricted assets. Our test can be applied to test portfolio efficiency while taking into account investments in non-traded labor income, housing and pension liabilities. We derive the exact small sample distribution of the test statistic under both the null hypothesis and the alternative hypothesis, under the assumption of a conditional multivariate normal distribution for the excess asset returns. The unrestricted GRS test is a special case within our framework. Simulation experiments demonstrate that our test performs well: the type I error of the test is very close to the desired significance level, while the asymptotic Wald test of Gouriéroux and Jouneau (1999) rejects the null too often in small samples (with 50 or 100 observations).

As an illustration, we apply our test to assess the mean-variance efficiency of a well-diversified US stock-bond portfolio for an individual investor with non-traded labor income. We use two sets of primitive test assets. The first set consists of 12 industry portfolios and the second set consist of four Fama and French size and value portfolios and two Ibbotson long-term bond portfolios. For the growth rate of the individual's labor income we use series “Average hourly earnings of production workers” in the US manufacturing sector. Exploiting the suitability of our test for small samples, we use 50 years of annual return data for the efficiency tests. In line with existing evidence, we find that mean-variance efficiency of the broad stock-bond portfolio cannot be rejected relative to the 12 industry portfolios, while efficiency is strongly rejected when size and value sorted portfolios are used as test assets. Taking into account the non-traded future labor income of the investor does not change the conclusions regarding portfolio efficiency, but it does considerably affect the magnitude, and even the sign, of the estimated alphas. For example, the estimated alpha of the Manufacturing industry portfolio changes from 0.3% per year to -3.4% per year, once we take labor income linked to average wage growth in the manufacturing sector into account.

Following GRS, our test assumes a serially-IID normal asset return distribution, without incorporating conditioning information about the state-of-the-world. Further research could focus on deriving a version of the test in a setting with conditioning information, following, for example, MacKinlay and Richardson (1991), Zhou (1993), Jagannathan and Wang (1996) and Ferson and Siegel (2006). Finally, we would like to stress that the mean-variance model can fail to distinguish between efficient and inefficient portfolios if the return distribution is not elliptical (see, for example, Chamberlain, 1983). To avoid possible specification error, we advise the empirical researcher to use mean-variance efficiency tests in combination with more general stochastic dominance efficiency tests.

Appendix

In this appendix we prove that the estimator of the generalized alphas defined in (10) follows a joint normal distribution conditional on the returns of the portfolios $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$. Let $\mathbf{R} \in \Re^{N\times T}$ denote a matrix containing the sample returns: $\mathbf{R} \equiv (\boldsymbol{r}_1 \cdots \boldsymbol{r}_T)$. Using $\hat{\boldsymbol{\mu}} \equiv T^{-1}\mathbf{R}\mathbf{1}_T$, $\tilde{\mathbf{R}} \equiv \mathbf{R} - \hat{\boldsymbol{\mu}}\mathbf{1}_T'$, $\tilde{\mathbf{R}}\tilde{\mathbf{R}}' = \mathbf{R}\tilde{\mathbf{R}}'$ and $\boldsymbol{\delta} \equiv (\tilde{\mathbf{R}}'\boldsymbol{\tau})(\boldsymbol{\tau}'\tilde{\mathbf{R}}\tilde{\mathbf{R}}'\boldsymbol{\tau})^{-1}$, we can write the OLS estimator for the betas in (7) as $\hat{\boldsymbol{\beta}} \equiv \mathbf{R}\boldsymbol{\delta}$. Further, it follows that $\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa} = \hat{\sigma}_{\boldsymbol{\tau\kappa}}\hat{\sigma}_{\boldsymbol{\tau}}^{-2}$, with $\hat{\sigma}_{\boldsymbol{\tau}}^2 \equiv (\boldsymbol{\tau}'\tilde{\mathbf{R}}\tilde{\mathbf{R}}'\boldsymbol{\tau})T^{-1}$ and $\hat{\sigma}_{\boldsymbol{\tau\kappa}} \equiv (\boldsymbol{\kappa}'\tilde{\mathbf{R}}\tilde{\mathbf{R}}'\boldsymbol{\tau})T^{-1}$. We can now write the estimator for the generalized alphas in (10) as:

$$\hat{\boldsymbol{\alpha}} = \hat{\boldsymbol{\mu}} - \hat{\boldsymbol{\beta}}(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa}) = \mathbf{R}\left(T^{-1}\mathbf{1}_T - \boldsymbol{\delta}(\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1})(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\right) \tag{A}$$

Reformulating the DGP in (8) in matrix notation as $\mathbf{R} = (\boldsymbol{\alpha}\mathbf{1}_T' + \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}'\mathbf{R} + \mathbf{U})$, with $\mathbf{U} \in \Re^{N\times T}$ denoting the matrix of regression errors $\mathbf{U} \equiv (\boldsymbol{u}_1 \cdots \boldsymbol{u}_T)$, we can now show that the generalized alphas are a linear function of the errors $\mathbf{U}$:

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \mathbf{U}\left(T^{-1}\mathbf{1}_T - \boldsymbol{\delta}(\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1})(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\right) \tag{B}$$

Proof of $\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + \mathbf{U}\left(T^{-1}\mathbf{1}_T - \boldsymbol{\delta}(\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1})(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\right)$:

Using $\mathbf{R} = (\boldsymbol{\alpha}\mathbf{1}_T' + \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}'\mathbf{R} + \mathbf{U})$, $\mathbf{1}_T'\boldsymbol{\delta} = 0$ and $\boldsymbol{\kappa}'\mathbf{R}\boldsymbol{\delta} = \boldsymbol{\kappa}'\hat{\boldsymbol{\beta}} = \hat{\sigma}_{\boldsymbol{\tau\kappa}}\hat{\sigma}_{\boldsymbol{\tau}}^{-2}$, we find

$$\begin{aligned}
\hat{\boldsymbol{\alpha}} &= \mathbf{R}(T^{-1}\mathbf{1}_T - (\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}\boldsymbol{\delta}) = (\boldsymbol{\alpha}\mathbf{1}_T' + \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}'\mathbf{R} + \mathbf{U})(T^{-1}\mathbf{1}_T - (\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}\boldsymbol{\delta}) \\
&= \boldsymbol{\alpha} - \boldsymbol{\alpha}(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}\mathbf{1}_T'\boldsymbol{\delta} + \boldsymbol{\beta}(\boldsymbol{\beta}'\boldsymbol{\kappa})^{-1}(T^{-1}\boldsymbol{\kappa}'\mathbf{R}\mathbf{1}_T - (\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}\boldsymbol{\kappa}'\mathbf{R}\boldsymbol{\delta}) \\
&\qquad + \mathbf{U}(T^{-1}\mathbf{1}_T - (\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})(\hat{\boldsymbol{\beta}}'\boldsymbol{\kappa})^{-1}\boldsymbol{\delta}) \\
&= \boldsymbol{\alpha} + \mathbf{U}\left(T^{-1}\mathbf{1}_T - (\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1}\boldsymbol{\delta}\right).
\end{aligned}$$

□

Conditional on the returns of the portfolios $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$, it follows from (B) that the estimator $\hat{\boldsymbol{\alpha}}$ follows a joint normal distribution with $E[\hat{\boldsymbol{\alpha}}]=\boldsymbol{\alpha}$. Below we provide the proof of $Var[\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}]=T^{-1}(1+\hat{\theta}^2)\boldsymbol{\Sigma}$.

Proof of $Var[\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}]=T^{-1}(1+\hat{\theta}^2)\boldsymbol{\Sigma}$:

Using $\boldsymbol{\delta}'\mathbf{1}_T=0$ and $\boldsymbol{\delta}'\boldsymbol{\delta}=(\boldsymbol{\tau}'\tilde{\mathbf{R}}\tilde{\mathbf{R}}'\boldsymbol{\tau})^{-1}=T^{-1}\hat{\sigma}_{\boldsymbol{\tau}}^{-2}$, we find

$$Var[\hat{\boldsymbol{\alpha}}-\boldsymbol{\alpha}]=(T^{-1}\mathbf{1}_T-(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1}\boldsymbol{\delta})(T^{-1}\mathbf{1}_T-(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1}\boldsymbol{\delta})\boldsymbol{\Sigma}$$

$$=(T^{-1}-2T^{-1}(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})\hat{\sigma}_{\boldsymbol{\tau}}^2\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1}\boldsymbol{\delta}'\mathbf{1}_T+(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})^2\hat{\sigma}_{\boldsymbol{\tau}}^4\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-2}\boldsymbol{\delta}'\boldsymbol{\delta})\boldsymbol{\Sigma}$$

$$=(T^{-1}+T^{-1}(\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa})^2(\hat{\sigma}_{\boldsymbol{\tau}}\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1})^2)\boldsymbol{\Sigma}$$

$$=T^{-1}(1+\hat{\theta}^2)\boldsymbol{\Sigma}\text{, with }\hat{\theta}=\hat{\boldsymbol{\mu}}'\boldsymbol{\kappa}(\hat{\sigma}_{\boldsymbol{\tau}}\hat{\sigma}_{\boldsymbol{\tau\kappa}}^{-1})=\hat{S}_{\kappa}\hat{\rho}_{\boldsymbol{\tau\kappa}}^{-1}$$

□

Proof of $\xi(\mathbf{A})=(1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}'\mathbf{M}(\mathbf{M}'\hat{\boldsymbol{\Sigma}}\mathbf{M})^{-1}\mathbf{M}'\hat{\boldsymbol{\alpha}}$:

The test statistic is the solution to an unrestricted minimization problem, that is, $\xi(\mathbf{A})=\min_{\boldsymbol{\rho}\in\Re^K}(1+\hat{\theta}^2)^{-1}(\hat{\boldsymbol{\alpha}}'-\boldsymbol{\rho}'\overline{\mathbf{A}})\hat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\alpha}}-\overline{\mathbf{A}}'\boldsymbol{\rho})$. The solution to this problem is $\boldsymbol{\rho}^*\equiv(\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{A}}')^{-}\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\alpha}}$ and $\xi(\mathbf{A})=(1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}'(\hat{\boldsymbol{\Sigma}}^{-1}-\hat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{A}}'(\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{A}}')^{-}\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1})\hat{\boldsymbol{\alpha}}$. Using Khatri's (1966) lemma, we find $(\hat{\boldsymbol{\Sigma}}^{-1}-\hat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{A}}'(\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1}\overline{\mathbf{A}}')^{-}\overline{\mathbf{A}}\hat{\boldsymbol{\Sigma}}^{-1})=\mathbf{M}(\mathbf{M}'\hat{\boldsymbol{\Sigma}}\mathbf{M})^{-1}\mathbf{M}'$ and thus $\xi(\mathbf{A})=(1+\hat{\theta}^2)^{-1}\hat{\boldsymbol{\alpha}}'\mathbf{M}(\mathbf{M}'\hat{\boldsymbol{\Sigma}}\mathbf{M})^{-1}\mathbf{M}'\hat{\boldsymbol{\alpha}}$.

□

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Table 1 Descriptive Statistics and Correlations of the Test Asset Returns – I

	T-bill	Long-term bonds		Stocks	Fama and French (2x3) size/value portfolios				Labor
	1-month	Corporate	Government	VW	Big/High	Big/Low	Small/High	Small/Low	Income
Mean	0.053	0.020	0.018	0.065	0.097	0.059	0.145	0.069	0.045
Median	0.051	0.032	0.002	0.098	0.145	0.062	0.156	0.051	0.036
Maximum	0.147	0.320	0.298	0.433	0.705	0.399	0.679	0.855	0.101
Minimum	0.010	-0.159	-0.152	-0.359	-0.310	-0.373	-0.344	-0.524	0.010
Std. Dev.	0.028	0.102	0.108	0.174	0.198	0.184	0.246	0.291	0.023
Skewness	0.990	0.404	0.549	-0.321	0.193	-0.198	0.080	0.460	0.882
Kurtosis	4.155	3.217	2.743	2.467	3.699	2.327	2.717	3.150	2.819
Jarque-Bera	10.942	1.458	2.646	1.450	1.329	1.270	0.220	1.811	6.558
p-value	0.004	0.482	0.266	0.484	0.514	0.530	0.896	0.404	0.038

Correlations	T-Bill 1m	Corp.bonds	Gov. bonds	Stocks VW	FF BH	FF BL	FF SH	FF SL	$g_{y,manu} - R_0$
T-Bill 1m (R_0)	1.000	-0.141	-0.094	-0.184	-0.142	-0.158	-0.166	-0.168	-0.654
Corp. Bonds	-0.141	1.000	0.950	0.231	0.307	0.213	0.221	0.004	-0.191
Gov. Bonds	-0.094	0.950	1.000	0.133	0.211	0.136	0.098	-0.099	-0.239
Stocks VW	-0.184	0.231	0.133	1.000	0.848	0.953	0.779	0.851	0.068
FF BH	-0.142	0.307	0.211	0.848	1.000	0.739	0.866	0.722	0.057
FF BL	-0.158	0.213	0.136	0.953	0.739	1.000	0.617	0.766	-0.003
FF SH	-0.166	0.221	0.098	0.779	0.866	0.617	1.000	0.840	0.165
FF SL	-0.168	0.004	-0.099	0.851	0.722	0.766	0.840	1.000	0.243
$g_{y,manu} - R_0$	-0.654	-0.191	-0.239	0.068	0.057	-0.003	0.165	0.243	1.000

The table shows descriptive statistics and correlations of the test asset returns used in Section 4 and 6, based on 50 annual return observations from 1956 to 2005. All asset returns are measured in excess of the 1-month T-Bill rate (source: Ibbotson and Associates). "Stock VW" denotes the value-weighted average of all common stocks listed on the NYSE, AMEX and NASDAQ markets and covered by CRSP. The returns for the long-term corporate and government bonds are based on total return indices from Ibbotson and Associates. We use four Fama and French portfolios, resulting from a 2x3 double sorting of stock based on size and value: small stocks with low price to book (SL), small stocks with high price to book (SH), big stocks with low price to book (BL) and big stocks with high price to book (BH). For the growth rate of labor income, denoted by $g_{y,manu}$, we use the yearly change in the series "Average hourly earnings of production workers" in the manufacturing sector from the US Bureau of Labor Statistics (http://www.bls.gov/). Descriptive statistics are reported for the original wage growth rate (without subtracting the risk-free rate), but for ease of comparison correlations with excess asset returns are based on wage growth rate in excess of the risk free.

Table 2 Descriptive Statistics and Correlations of the Test Asset Returns – II

	Business equipment	Chemicals	Consumer durables	Energy	Healthcare and drugs	Manufact.	Finance	Cons. non-durables	Other industries	Wholesale and retail	Telecom	Utilities
Mean	0.090	0.056	0.063	0.082	0.095	0.061	0.085	0.092	0.068	0.082	0.065	0.062
Median	0.117	0.072	0.060	0.103	0.078	0.096	0.072	0.081	0.114	0.074	0.039	0.070
Maximum	0.797	0.352	0.679	0.561	0.579	0.503	0.469	0.491	0.459	0.614	0.495	0.454
Minimum	-0.455	-0.267	-0.481	-0.361	-0.295	-0.385	-0.409	-0.351	-0.406	-0.433	-0.421	-0.291
Std. Dev.	0.280	0.162	0.252	0.194	0.205	0.186	0.205	0.189	0.201	0.230	0.197	0.171
Skewness	0.211	-0.276	0.222	-0.156	0.290	-0.051	-0.015	-0.103	-0.425	0.176	-0.062	0.013
Kurtosis	2.990	2.176	2.594	3.119	2.651	2.886	2.646	2.929	2.296	2.905	2.715	2.650
Jarque-Bera	0.372	2.051	0.755	0.231	0.956	0.048	0.263	0.099	2.538	0.276	0.201	0.257
p-value	0.830	0.359	0.685	0.891	0.620	0.976	0.877	0.952	0.281	0.871	0.904	0.879

Correlations	BUSEQ	CHEMS	DURBL	ENRGY	HLTH	MANUF	MONEY	NODUR	OTHER	SHOPS	TELCM	UTILS
BUSEQ	1.000	0.598	0.597	0.308	0.539	0.721	0.505	0.453	0.738	0.675	0.577	0.262
CHEMS	0.598	1.000	0.724	0.552	0.611	0.840	0.696	0.733	0.815	0.777	0.604	0.510
DURBL	0.597	0.724	1.000	0.389	0.376	0.755	0.630	0.635	0.756	0.816	0.577	0.573
ENRGY	0.308	0.552	0.389	1.000	0.331	0.656	0.529	0.392	0.673	0.351	0.260	0.555
HLTH	0.539	0.611	0.376	0.331	1.000	0.592	0.664	0.729	0.626	0.646	0.460	0.579
MANUF	0.721	0.840	0.755	0.656	0.592	1.000	0.740	0.734	0.888	0.760	0.488	0.582
MONEY	0.505	0.696	0.630	0.529	0.664	0.740	1.000	0.825	0.767	0.759	0.577	0.737
NODUR	0.453	0.733	0.635	0.392	0.729	0.734	0.825	1.000	0.666	0.837	0.560	0.770
OTHER	0.738	0.815	0.756	0.673	0.626	0.888	0.767	0.666	1.000	0.804	0.638	0.598
SHOPS	0.675	0.777	0.816	0.351	0.646	0.760	0.759	0.837	0.804	1.000	0.644	0.639
TELCM	0.577	0.604	0.577	0.260	0.460	0.488	0.577	0.560	0.638	0.644	1.000	0.515
UTILS	0.262	0.510	0.573	0.555	0.579	0.582	0.737	0.770	0.598	0.639	0.515	1.000
T-Bill 1m (R_0)	-0.265	-0.205	-0.243	-0.140	-0.071	-0.246	-0.117	-0.036	-0.135	-0.106	-0.033	-0.114
Corp. bonds	-0.016	0.287	0.318	0.006	0.236	0.127	0.374	0.421	0.198	0.347	0.303	0.506
Gov. bonds	-0.099	0.167	0.175	-0.052	0.249	0.024	0.316	0.343	0.110	0.220	0.261	0.480
Stocks VW	0.789	0.872	0.773	0.660	0.690	0.906	0.817	0.762	0.950	0.860	0.722	0.654
$g_{y,manu} - R_0$	0.175	0.015	0.136	0.158	-0.117	0.188	-0.002	-0.128	0.056	-0.038	-0.129	-0.081

The table shows descriptive statistics and correlations of the industry returns used in Section 6, based on 50 annual return observations from 1956 to 2005. The 12 industry portfolio returns are from the data library of Kenneth French, value-weighted and measured in excess of the 1-month T-Bill rate. "Stock VW" denotes the CRISP value-weighted average of all US common stocks. Bond returns are from Ibbotson and Associates. The growth rate of labor income, denoted by $g_{y,manu}$, is the yearly change in "Average hourly earnings of production workers" in the manufacturing sector from the US Bureau of Labor Statistics (http://www.bls.gov/).

Table 3 Multivariate Normal Simulation Results

	Statistic	Mean	Var.	Size at signif. level 10.0%	5.0%	1.0%	Power at signif. level 10.0%	5.0%	1.0%
Unconstrained	GRS *F*	1.05	0.44	10.0%	5.1%	1.0%	96.9%	93.6%	80.0%
T = 50	F(6,43)	1.05	0.44						
S = 100,000	JK χ^2	8.74	27.18	21.3%	13.8%	5.2%	98.8%	97.6%	92.8%
	$\chi^2(7)$	7.00	14.00						
Constrained	KP *F*	1.04	0.61	9.5%	4.7%	1.0%	85.1%	76.9%	55.6%
T = 50	F(4,45)	1.05	0.63						
S = 100,000	GJ χ^2	6.01	17.40	17.7%	10.7%	3.8%	88.0%	81.5%	65.3%
	$\chi^2(5)$	5.00	10.00						
Unconstrained	GRS *F*	1.02	0.38	10.0%	4.9%	1.0%	90.3%	83.4%	62.9%
T = 100	F(6,93)	1.02	0.38						
S = 50,000	JK χ^2	7.76	19.04	15.1%	8.7%	2.5%	94.9%	90.9%	78.2%
	$\chi^2(7)$	7.00	14.00						
Constrained	KP *F*	1.01	0.55	9.8%	5.0%	1.0%	90.1%	83.1%	63.1%
T = 100	F(4,95)	1.02	0.56						
S = 50,000	GJ χ^2	5.44	13.01	13.6%	7.7%	2.0%	92.8%	87.5%	72.2%
	$\chi^2(5)$	5.00	10.00						
Unconstrained	GRS *F*	1.01	0.35	10.0%	4.9%	1.0%	99.8%	99.4%	97.3%
T = 200	F(6,193)	1.01	0.35						
S = 25,000	JK χ^2	7.35	16.23	12.4%	6.8%	1.6%	99.9%	99.8%	98.9%
	$\chi^2(7)$	7.00	14.00						
Constrained	KP *F*	1.00	0.52	9.7%	4.9%	0.9%	99.7%	99.2%	96.4%
T = 200	F(4,195)	1.01	0.53						
S = 25,000	GJ χ^2	5.20	11.21	11.5%	6.2%	1.4%	99.8%	99.5%	97.9%
	$\chi^2(5)$	5.00	10.00						
Unconstrained	GRS *F*	1.00	0.33	9.6%	4.7%	0.9%	99.8%	99.6%	98.2%
T = 400	F(6,393)	1.01	0.34						
S = 12,500	JK χ^2	7.14	14.78	11.0%	5.6%	1.1%	99.9%	99.8%	99.0%
	$\chi^2(7)$	7.00	14.00						
Constrained	KP *F*	1.00	0.49	9.8%	4.6%	0.9%	99.8%	99.5%	98.2%
T = 400	F(4,395)	1.01	0.51						
S = 12,500	GJ χ^2	5.08	10.42	10.8%	5.3%	1.0%	99.9%	99.7%	98.5%
	$\chi^2(5)$	5.00	10.00						
Unconstrained	GRS *F*	1.01	0.34	9.9%	5.0%	1.1%	100.0%	100.0%	100.0%
T = 800	F(6,793)	1.00	0.34						
S = 6,250	JK χ^2	7.14	14.87	10.8%	5.4%	1.3%	100.0%	100.0%	100.0%
	$\chi^2(7)$	7.00	14.00						
Constrained	KP *F*	1.01	0.51	10.4%	5.1%	1.0%	100.0%	100.0%	100.0%
T = 800	F(4,795)	1.00	0.51						
S = 6,250	GJ χ^2	5.11	10.44	10.7%	5.5%	1.1%	100.0%	100.0%	100.0%
	$\chi^2(5)$	5.00	10.00						

The label GRS *F* denotes the test for mean-variance efficency of Gibbons, Ross and Shanken (1989), while JK χ^2 denotes the Wald test of Jobson and Korkie (1982). KP *F* denotes the test for mean-variance efficiency under portfolio weight constraints derived in this paper, while GJ χ^2 denotes the Wald test under constraints of Gouriéroux and Jouneau (1999). The theoretical mean and variance of the test statistic under the null hypothesis are shown below each row of results. The size estimates are based on the percentage of simulation rounds in which the efficiency of the ex post tangency portfolio was rejected, while the power is based on the percentage of simulation rounds in which the efficiency of an equally weighted portfolio was rejected.

Table 4 Mean-Variance Efficiency Test Results – I

	Unrestricted	With labor income		
	GRS	$Y_t/W_t = 0.5$	$Y_t/W_t = 0.7$	$Y_t/W_t = 0.9$
Portfolio	$\hat{\alpha}_i$	$\hat{\alpha}_i$	$\hat{\alpha}_i$	$\hat{\alpha}_i$
BUSEQ	0.026	0.020	0.012	-0.031
CHEMS	0.006	0.005	0.004	-0.003
DURBL	-0.008	-0.012	-0.018	-0.051
ENRGY	0.044	0.040	0.036	0.009
HLTH	0.043	0.044	0.047	0.059
MANUF	0.007	0.003	-0.003	-0.034
MONEY	0.021	0.020	0.019	0.013
NODUR	0.034	0.036	0.038	0.051
OTHER	0.003	0.002	-0.001	-0.016
SHOPS	0.010	0.010	0.009	0.009
TELCM	0.011	0.013	0.016	0.029
UTILS	0.012	0.013	0.014	0.019
$\sqrt{\sum \hat{\alpha}_i^2}$	0.081	0.079	0.079	0.112
ξ	1.235	1.244	1.243	0.839
p-value	0.297	0.292	0.293	0.612

The table shows the results of the mean-variance efficiency test for a proxy of the market portfolio, consisting of 50% US stocks (value-weighted, CRISP), 25% long-term US government bonds (Ibbotson and Associates) and 25% long-term US corporate bonds (Ibbotson and Associates), relative to the returns of 12 industry portfolios (value-weighted, from the data library of Kenneth French). Efficiency test results are presented for an unrestricted investor without labor income (GRS) and for investors with non-traded labor income with the weight of the net present value of labor income fixed at Y_t/W_t = 0.5, 0.7 and 0.9, respectively, relative to total wealth. The growth rate of labor income is the yearly change in "Average hourly earnings of production workers" in the manufacturing sector from the US Bureau of Labor Statistics. The table show estimated alphas of the 12 industry portfolios, the average deviation of the alphas from the null hypothesis, the value of the test statistic (ξ) and the corresponding p-value.

Table 5 Mean-Variance Efficiency Test Results – II

	Unrestricted	With labor income		
	GRS	$Y_t/W_t = 0.5$	$Y_t/W_t = 0.7$	$Y_t/W_t = 0.9$
Portfolio	$\hat{\alpha}_i$	$\hat{\alpha}_i$	$\hat{\alpha}_i$	$\hat{\alpha}_i$
Gov. bonds	-0.006	-0.004	-0.001	0.016
Corp. bonds	-0.005	-0.004	-0.002	0.010
FF BH	0.036	0.034	0.032	0.017
FF BL	-0.001	-0.002	-0.003	-0.008
FF SH	0.079	0.074	0.067	0.030
FF SL	-0.004	-0.012	-0.023	-0.083
$\sqrt{\sum \hat{\alpha}_i^2}$	0.087	0.083	0.078	0.092
ξ	4.451	4.519	4.501	2.874
p-value	0.001	0.001	0.001	0.019

The table shows the results of the mean-variance efficiency test for a proxy of the market portfolio, consisting of 50% US stocks (value-weighted, CRISP), 25% long-term US government bonds (Ibbotson and Associates) and 25% long-term US corporate bonds (Ibbotson and Associates), relative to the returns of a portfolio of long-term corporate bonds, long-term government bonds and the returns of four Fama and French portfolios. The Fama and French portfolios are the result of a 2x3 double sorting of stock based on size and value: small stocks with low price to book (SL), small stocks with high price to book (SH), big stocks with low price to book (BL) and big stocks with high price to book (BH). Efficiency test results are presented for an unrestricted investor without labor income (GRS) and for investors with non-traded labor income with the weight of the net present value of labor income fixed at $Y_t/W_t = 0.5$, 0.7 and 0.9, respectively, relative to total wealth. The growth rate of labor income is the yearly change in "Average hourly earnings of production workers" in the manufacturing sector from the US Bureau of Labor Statistics. The table show estimated alphas for the 6 test asset portfolios, the average deviation of the alphas from the null hypothesis, the value of the test statistic (ξ) and the corresponding p-value.

Footnotes

[1] In practice, Treasury securities (T-bills, T-bonds and TIPS) promise riskless yields to maturity and can serve as riskless assets. The availability of these assets in the market of course does not exclude restrictions on riskless lending and borrowing. We use the riskless asset primarily to construct excess returns for the risky assets. Excess returns typically are less sensitive to time-variation than nominal returns. Also, if excess returns are used, then there is no need for an explicit budget restriction in the portfolio possibilities set.

[2] The KKT conditions also include a feasibility condition, in this case $\mathbf{A}\boldsymbol{\tau}_2 = \mathbf{b}$. Please note that this condition is always satisfied trivially, as $\boldsymbol{\tau} \in \Lambda$ by definition.

[3] GRS do not explicitly specify a mean-variance objective function and therefore do not choose a specific risk aversion parameter. However, $\zeta = (\boldsymbol{\mu}'\boldsymbol{\tau})(\boldsymbol{\tau}'\boldsymbol{\Omega}\boldsymbol{\tau})^{-1}$ is the only level of risk aversion that makes the GRS approach consistent with the maximization of a mean-variance objective function.

[4] In the unrestricted case, $\boldsymbol{\kappa} = \boldsymbol{\tau}$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}_{GRS}$. Further, equation (8) reduces to (7), i.e. $\boldsymbol{r}_t = \boldsymbol{\alpha}_{GRS} + \boldsymbol{\beta}(\mathbf{r}_t'\boldsymbol{\tau}) + \boldsymbol{\varepsilon}_t$, $t = 1, \cdots, T$, while $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\varepsilon$ and $\hat{\theta} = \hat{S}_\tau$. Thus, our framework includes the unrestricted GRS test a special case. In the GRS approach, the return distribution is assumed to be normal given the returns of the portfolio $\boldsymbol{\tau}$. Effectively, the estimated Sharpe ratio $\hat{S}_{\boldsymbol{\tau}}$ of the evaluated portfolio $\boldsymbol{\tau}$ is treated as non-stochastic while deriving the return distribution of the estimated alphas $\hat{\boldsymbol{\alpha}}_{GRS}$. In our approach, the return distribution is assumed to be normal given the returns of the entire portfolio $\boldsymbol{\tau}$ and the returns of the unrestricted asset portfolio $\boldsymbol{\kappa}$. Effectively, the empirical Sharpe ratio $\hat{S}_{\boldsymbol{\tau}}$ of portfolio $\boldsymbol{\kappa}$ and the estimated correlation coefficient $\hat{\rho}_{\boldsymbol{\tau\kappa}}$ are treated as non-stochastic while deriving the return distribution of the estimated generalized alphas $\hat{\boldsymbol{\alpha}}$.

[5] For GRS and our test knowledge of the historical returns of the unconstrained asset portfolio and the constrained portfolio are sufficient, without the need to explicitly specify the portfolio weights of the primary assets (which are typically unknown in most asset pricing applications).

[6] Our framework can take into account additional restricted investments in housing. For ease of exposition we ignore homeownership here, e.g. assuming that the investor rents an apartment.